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PRODUCTS OF PARACOMPACT AND NORMAL SPACES

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled PRODUCTS OF PARACOMPACT AND NORMAL SPACES, submitted by LORNE LINDENBERG in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

A question which has concerned topologists for sometime is the following: since the product of even two paracompact or normal spaces may fail to be paracompact or normal, what properties must these spaces have in order for the product to be paracompact or normal? In this thesis, we report the best of the known results, simplifying the proofs whenever possible by assuming that all spaces are Hausdorff.

Chapter I is devoted to a study of paracompactness and normality in small products. Specifically, we are interested in those spaces X which have a paracompact or normal product $X \times Y$ with every space having the property $\mathcal{P}$. In §2, we assume that Y is a metric space and prove a result due to K. Morita, namely that $X \times Y$ is normal for every metric space Y if and only if X is a normal P - space. In §3, Y is any paracompact space, and we examine three independent results due to K. Morita, Y. Katuta, and T. Ishii. In §4, we present results for the case when Y is a compact metric space and include a brief discussion of Dowker's conjecture. In §5, compact spaces Y are dealt with using Tamano's famous theorem which states that X is paracompact if and only if $X \times \beta X$ is normal.

Paracompactness and normality in larger products are considered in Chapter II. In §7 we prove three theorems which express conditions under which a countable product of paracompact spaces will be paracompact. §8 gives a similar theorem for normal products. The most important part of

Chapter II is §9. In this section, the unpredictability of higher powers of spaces is demonstrated by examining some examples due to E. Michael.

While the major portion of our thesis is devoted to paracompactness and normality, Chapter III provides some results for other properties in product spaces. The properties considered are the Lindelöf property (§11), countable compactness (§12), pseudocompactness (§13), and perfect normality (§14).

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CHAPTER I

Paracompactness and Normality in Small Products

§1. Introduction.

1.1. In 1948, R. Sorgenfrey gave an example, now well known, of a paracompact space S such that the product space $S \times S$ is not paracompact or normal. The space S, known as the Sorgenfrey line, consists of the real line with the topology which has as a base all half open intervals of the form $[a,b)$ where a and b are rational and $a < b$.

It is then natural to ask the following question: given two topological spaces, X and Y, what restrictions must be placed on these spaces in order that the product space, $X \times Y$, be paracompact or normal? This question has been studied extensively in the last decade, a substantial portion of the work having been done by K. Morita, H. Tamano, and E. Michael. However, the known results are by no means complete as many questions have yet to be resolved.

The purpose of this chapter is to present and simplify, where possible, some of the known results. One of the major tasks will be to give a detailed and organized account of Morita's work. Specifically, Morita answers the question: which space X have a paracompact or normal product with every metric space Y? The remaining sections in this chapter will be devoted to answering the same question for some other class of spaces Y. In some cases, necessary and sufficient conditions can be given for the space X. In others, only sufficient conditions are available.

We conclude this section with the definitions and some important results concerning paracompact and normal spaces. These will form the foundation for our exposition. Note that, unless otherwise specified, all spaces will be assumed to be Hausdorff. This assumption will permit simplification of some of the results.

1.2. Review of Normality and Paracompactness. A topological space X is said to be NORMAL if for every pair of disjoint closed subsets G and H of X, there exist disjoint open subsets U and V of X such that $U \supset G$ and $V \supset H$. The elementary properties of normal spaces, for example, the preservation of normality in closed subspaces or closed continuous images, can be found developed in any text on general topology ([W], p. 100).

A topological space X is said to be PARACOMPACT if every open cover of X has an open locally finite refinement. Recall that if $\mathcal{U}$ and $\mathcal{V}$ are covers of a space X, then the cover $\mathcal{U}$ is said to REFINE the cover $\mathcal{V}$ if each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. A cover $\mathcal{U}$ of the space X is said to be LOCALLY FINITE if each $x \in X$ has a neighborhood V_x meeting only finitely many $U \in \mathcal{U}$. Some important facts about paracompact spaces are that an F_σ - subset (a subset which can be written as a countable union of closed sets) and the closed continuous image of a paracompact space are paracompact. Also, the topological product of a compact space[1] and a paracompact space is paracompact ([W], p. 148).

1. A space X is said to be COMPACT if every open cover of X has a finite subcover.

One important result to which we will often refer is that every paracompact space is normal ([W], p. 147). Of course, it is not true that every normal space is paracompact. As a counterexample, consider the space Ω_o of countable ordinals with the order topology. Ω_o is normal but not paracompact ([W], p. 148). Finally, reference will be made to a theorem due to A.H. Stone which states that every metric space is paracompact ([W], p. 147).

§2. Metric Spaces.

2.1. In this section we will study in depth Morita's solution to the problem: what spaces X have a normal product with every metric space Y ? Morita has done a great deal of work on this problem and to our knowledge, a concise account of his research does not exist.

Morita's major result is that the product $X \times Y$ is normal for every metric space Y if and only if X is a normal P-space. Before proving this theorem, we will have to discuss in some detail several concepts and establish a number of other results. We have been able to condense some of Morita's work since he often proves more than is necessary.

2.2. P-Spaces. The basis of Morita's study of normal products is a class of spaces known as P-spaces. A topological space X is said to be a P-SPACE if for any set Ω of indices and for any family $\{G(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \ ; \ i = 1,2,\cdots\}$ of open subsets of X satisfying the condition

(1) $G(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$ for $\alpha_1,\cdots,\alpha_{i+1} \in \Omega$, $i = 1,2,\cdots$

there exists a family $\{F(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \quad \Omega \ ; \ i = 1,2,\cdots\}$ of closed subsets of X satisfying the following two conditions:

(2) $\quad F(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i)$ for $\alpha_1,\cdots,\alpha_i \in \Omega$, and

(3) $\quad X = \bigcup_{i=1}^{\infty} F(\alpha_1,\cdots,\alpha_i)$ for any sequence (α_i) such that

$$X = \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i) \quad .$$

<u>Examples</u>. (a) The "scattered line" ([W], p. 40) is an example of a space which is not a P-space.

(b) It is easy to verify that X is a P-space in each of the following cases: (i) X is paracompact and perfectly normal[2], (ii) X is paracompact Hausdorff and topologically complete in the sense of E. Cech[3], (iii) X is countably compact[4] and normal, (iv) X is σ-compact[5] and regular[6] Hausdorff.

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2. A T_1 - space X is said to be PERFECTLY NORMAL if for each pair of disjoint closed sets A and B in X , there is a continuous function $f : X \to [0,1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

3. A completely regular space X is said to be TOPOLOGICALLY COMPLETE IN THE SENSE OF E. CECH if X is a G_δ - subset (a subset which can be written as a countable intersection of open sets) of its Stone-Cech compactification.

4. A topological space X is said to be COUNTABLY COMPACT if every countable open cover has a finite subcover.

5. A topological space X is σ-COMPACT if it is a countable union of compact subsets.

The following lemma provides us with some equivalent conditions which may be used in our definition of P-space.

Lemma 2.1. Let $\{G(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \; ; \; i = 1,2,\cdots\}$ be a family of open subsets of X satisfying condition (1) above. Then the following statements are equivalent for a normal space X .

(a) There exists a family $\{F(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \; ; \; i = 1,2,\cdots\}$ of closed subsets of X satisfying (2) and (3).

(b) There exists a family $\{F(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \; ; \; i = 1,2,\cdots\}$ of F_σ - subsets of X satisfying (2) and (3).

(c) There exists a family $\{F(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \; ; \; i = 1,2,\cdots\}$ of open F_σ - subsets of X satisfying (2) and (3).

Proof: (a) $\to$ (c) : Assume that (a) holds. Then for every sequence of elements (α_i) from Ω we have a closed set $F(\alpha_1,\cdots,\alpha_i)$ such that $F(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i)$. Since X is normal, there exists an open F_σ - subset $H(\alpha_1,\cdots,\alpha_i)$ of X such that $F(\alpha_1,\cdots,\alpha_i) \subset H(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i)$. Conditions (2) and (3) obviously hold for these sets $H(\alpha_1,\cdots,\alpha_i)$. Hence (c) is valid.

(c) $\to$ (b) : The is obvious.

(b) $\to$ (a) : Assume that (b) is true. Since $F(\alpha_1,\cdots,\alpha_i)$ is an F_σ - subset of X , there exist closed subsets $C(\alpha_1,\cdots,\alpha_i;k)$ of X

6. A space X is said to be REGULAR if for any point $x \in X$ and any closed set B such that $x \notin B$, there exist disjoint open sets U and V such that $U \ni x$ and $V \supset B$.

such that $F(\alpha_1,\cdots,\alpha_i) = \bigcup_{k=1}^{\infty} C(\alpha_1,\cdots,\alpha_i;k)$. Let us put

$$K(\alpha_1,\cdots,\alpha_i) = \cup\{C(\alpha_1,\cdots,\alpha_j;k) \mid j \leq i , k \leq i\} .$$

Then $K(\alpha_1,\cdots,\alpha_i)$ is closed in X and $K(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i)$, since for $j \leq i$, $C(\alpha_1,\cdots,\alpha_j;k) \subset F(\alpha_1,\cdots,\alpha_j) \subset G(\alpha_1,\cdots,\alpha_j) \subset G(\alpha_1,\cdots,\alpha_i)$. Suppose that $X = \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i)$ for some sequence (α_i) . Then by assumption, $X = \bigcup_{i=1}^{\infty} F(\alpha_1,\cdots,\alpha_i)$. But each $C(\alpha_1,\cdots,\alpha_j;k) \subset K(\alpha_1,\cdots,\alpha_i)$ for some i such that $i \geq j$, $i \geq k$. Therefore $\bigcup_{i=1}^{\infty} K(\alpha_1,\cdots,\alpha_i) = X$. Hence (a) is valid.

We can make our definition of a P-space a little more specific. If m is a cardinal number ≥ 1 , then a topological space X is said to be a P(m) - SPACE if for a set Ω of power m and for any family $\{G(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega ; i = 1,2,\cdots\}$ of open subsets of X such that

(1) $G(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$ for $\alpha_1,\cdots,\alpha_{i+1} \in \Omega$; $i = 1,2,\cdots$ there exists a family $\{F(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega ; i = 1,2,\cdots\}$ of closed subsets of X satisfying the following two conditions:

(2) $F(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i)$ for $\alpha_1,\cdots,\alpha_i \in \Omega$; $i = 1,2,\cdots$, and

(3) $X = \bigcup_{i=1}^{\infty} F(\alpha_1,\cdots,\alpha_i)$ for any sequence (α_i) such that

$$X = \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i) .$$

In comparing this definition to our definition of a P-space, it is apparent that the only difference between a P-space and a $P(m)$ - space is the cardinality of the index set. It is thus obvious that X is a P-space if and only if X is a $P(m)$ - space for each cardinal number m.

The following results exhibit some important properties of $P(m)$ - spaces. The proof of Theorem 2.2 is obvious.

Theorem 2.2. If $m > n$, then any $P(m)$ - space is a $P(n)$ - space.

Theorem 2.3. A normal space X is a $P(1)$ - space if and only if it is countably paracompact[7].

Proof: It is well known ([W], p. 156) that a normal space is countably paracompact if and only if whenever (G_i) is an increasing sequence of open sets with $\bigcup_i G_i = X$, then (G_i) can be "contracted" to a sequence (F_i) of closed sets with $F_i \subset G_i$ and $\bigcup F_i = X$. If $\Omega = \{\alpha\}$, the only sets available as $G(\alpha_1,\cdots,\alpha_i)$ in the definition of $P(1)$ - spaces are $G(\alpha) = G_1$, $G(\alpha,\alpha) = G_2$, and so on. The definition of $P(1)$ - space in this notation becomes the above characterization of countable paracompactness.

Theorem 2.4. Any normal $P(m)$ - space is countably paracompact.

Proof: This follows immediately from Theorems 2.2 and 2.3.

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7. A space is COUNTABLY PARACOMPACT if and only if every countable open cover has a locally finite refinement. (See [W], p. 156 for equivalent definitions of countable paracompactness.)

Theorem 2.5. Let f be a closed continuous mapping from a space X onto a space Y. If Y is a $P(m)$ - space and if $f^{-1}(y)$ is countably compact for each point $y \in Y$, then X is also a $P(m)$ - space.

Proof: Let $\{G(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \ ; \ i = 1,2,\cdots\}$ be a family of open subsets of X such that $G(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$ for $\alpha_1,\cdots,\alpha_i \in \Omega$ where $|\Omega| = m$. Let us put

$$H(\alpha_1,\cdots,\alpha_i) = Y - f(X-G(\alpha_1,\cdots,\alpha_i)) \ .$$

Then each $H(\alpha_1,\cdots,\alpha_i)$ is open in Y and clearly $H(\alpha_1,\cdots,\alpha_i) \subset H(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$ for $\alpha_1,\cdots,\alpha_i,\alpha_{i+1} \in \Omega$. We shall show that $Y = \bigcup_{i=1}^{\infty} H(\alpha_1,\cdots,\alpha_i)$ whenever $X = \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i)$. For suppose there exists $y_o \in Y$ such that $y_o \in Y - \bigcup_{i=1}^{\infty} H(\alpha_1,\cdots,\alpha_i)$. Then we have that $f^{-1}(y_o) \cap (X - G(\alpha_1,\cdots,\alpha_i)) \neq \phi$ for $i = 1,2,\cdots\}$, and hence the family $\{f^{-1}(y_o) \cap (X - G(\alpha_1,\cdots,\alpha_i)) \mid i = 1,2,\cdots\}$ has the finite intersection property. Since $f^{-1}(y_o)$ is countably compact, we have

$$\bigcap_{i=1}^{\infty} (X - G(\alpha_1,\cdots,\alpha_i)) \neq \phi$$

which contradicts our assumption that $X = \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i)$. Hence $Y = \bigcup_{i=1}^{\infty} H(\alpha_1,\cdots,\alpha_i)$.

Since Y is a $P(m)$ - space, there exists a family $\{K(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \ ; \ i = 1,2,\cdots\}$ of closed subsets of Y

such that $K(\alpha_1,\cdots,\alpha_i) \subset H(\alpha_1,\cdots,\alpha_i)$ and $Y = \bigcup_{i=1}^{\infty} K(\alpha_1,\cdots,\alpha_i)$ whenever $Y = \bigcup_{i=1}^{\infty} H(\alpha_1,\cdots,\alpha_i)$. Let us put $F(\alpha_1,\cdots,\alpha_i) = f^{-1}(K(\alpha_1,\cdots,\alpha_i))$. Then each $F(\alpha_1,\cdots,\alpha_i)$ is a closed subset of X, and we have $F(\alpha_1,\cdots,\alpha_i) \subset f^{-1}(H(\alpha_1,\cdots,\alpha_i) = X - f^{-1}[f(X-G(\alpha_1,\cdots,\alpha_i))] \subset G(\alpha_1,\cdots,\alpha_i)$. Moreover, if $X = \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i)$, then $Y = \bigcup_{i=1}^{\infty} H(\alpha_1,\cdots,\alpha_i)$ and consequently $X = \bigcup_{i=1}^{\infty} F(\alpha_1,\cdots,\alpha_i)$. Hence X is a $P(m)$ - space.

Corollary. If X is a $P(m)$ - space and Q is a compact space, then the product space $X \times Q$ is a $P(m)$ - space.

Proof: The projection from $X \times Q$ onto X is a closed continuous mapping since Q is compact. Hence Theorem 2.5 applies.

2.3. Normal Coverings. Let X be a topological space. An open covering $\mathcal{U}$ of X is said to be a STAR-REFINEMENT of another open covering $\mathcal{V}$ if the covering $\{St.(U,\mathcal{U}) \mid U \in \mathcal{U}\}$ is a refinement of $\mathcal{V}$, where $St.(U,\mathcal{U}) = \cup\{A \in \mathcal{U} \mid U \cap A \neq \phi\}$. A sequence $\{\mathcal{U}_i \mid i = 1,2,\cdots\}$ of open coverings of X is said to be NORMAL if $\mathcal{U}_{i+1}$ is a star-refinement of $\mathcal{U}_i$ for $i = 1,2,\cdots$. Finally, an open covering $\mathcal{U}$ of X is said to be a NORMAL COVERING if there is a normal sequence $\{\mathcal{U}_i \mid i = 1,2,\cdots\}$ of open coverings of X such that $\mathcal{U}_1$ is a refinement of $\mathcal{U}$.

Normal coverings also play an important role in Morita's study of normal products. The following result concerning normal covers will be needed to prove the main result.

<u>Lemma 2.6.</u> Let $\mathcal{G} = \{G_\alpha \mid \alpha \in \Omega\}$ be a normal covering of a space X. Then $\mathcal{G}\times Y = \{G_\alpha \times Y \mid \alpha \in \Omega\}$ is a normal covering of $X\times Y$ for any topological space Y.

<u>Proof</u>: Let $\{\mathcal{G}_i \mid i = 1,2,\cdots\}$ be a normal sequence of open covers of X such that $\mathcal{G}_1$ is a refinement of $\mathcal{G}$. Consider $\mathcal{H}_i = \{G\times Y \mid G \in \mathcal{G}_i\}$ for $i = 1,2,\cdots$. Then for any subset A of X, we have that $St.(A\times Y,\mathcal{H}_i) = St.(A,\mathcal{G}_i) \times Y$. Therefore, $\mathcal{H}_{i+1}$ is a star-refinement of $\mathcal{H}_i$ for $i = 1,2,\cdots$. But then $\{\mathcal{H}_i \mid i = 1,2,\cdots\}$ is a normal sequence of open coverings of $X\times Y$ such that $\mathcal{H}_1$ is a refinement of $\mathcal{G}\times Y$. Hence $\mathcal{G}\times Y$ is a normal covering of $X\times Y$.

<u>Lemma 2.7.</u> Every σ - locally finite open covering of a countably paracompact and normal space is normal.

<u>Proof</u>: Let $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ be a σ - locally finite open covering of a countably paracompact and normal space X. Then $\mathcal{G}_i$ is locally finite for each i. Let $G_i = \cup\{G \mid G \in \mathcal{G}_i\}$. Then $\{G_i \mid i = 1,2,\cdots\}$ is a countable open covering of X, and since X is countably paracompact, there exists a locally finite countable open covering $\{H_i\}$ of X such that $H_i \subset G_i$ for $i = 1,2,\cdots$. Then $\{H_i \cap G \mid G \in \mathcal{G}_i , i = 1,2,\cdots\}$ is a locally finite open covering of X, and hence a normal covering of X (every locally finite open cover of a normal space is a normal cover), and is a refinement of $\mathcal{G}_i$. Hence, $\mathcal{G}$ is a normal covering of X.

The following theorem provides us with a valuable tool which we will use often in establishing other results. It gives a number of conditions which are equivalent to a cover of a space being a normal cover.

Theorem 2.8. Let G be an open covering of a space X. Then the following are equivalent:

(a) G is a normal cover;

(b) there exists a continuous function f from X into a metric space Y such that G is refined by the inverse image of some open cover of Y;

(c) G admits a locally finite open normal covering as its refinement;

(d) G admits as its refinement a locally finite open covering $\{H_\lambda\}$, each set of which is expressed as $H_\lambda = \{x \mid f_\lambda(x) > 0\}$ where $f_\lambda : X \to [0,1]$ is a continuous function. H_λ is called a cozero set;

(e) G has a locally finite partition of unity subordinated to it[8];

(f) G has a partition of unity subordinated to it;

(g) there exists a normal open covering $\{U_\gamma\}$ of X such that on each of the subspaces U_γ, the covering $\{G \cap U_\gamma \mid G \in G\}$ is normal.

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8. A PARTITION OF UNITY is a collection ϕ of continuous functions from X to the non-negative reals such that, at each $x \in X$, $\psi(x) \neq 0$ for only finitely many $\psi \in \phi$, and $\sum_{\psi \in \phi} \psi(x) = 1$. ϕ is called LOCALLY FINITE if and only if each $x \in X$ has a neighborhood on which all but finitely many $\psi \in \phi$ vanish. ϕ is SUBORDINATED to a cover U of X if and only if each $\psi \in \phi$ vanishes outside some $U \in U$.

Proof: (a) → (b) . This is well known ([W], p. 167).

(b) → (a) . Let $\mathcal{G}$ be an open cover of X . By assumption, there exists a continuous map $f : X \to Y$, a metric space, such that $\mathcal{G}$ is refined by the inverse image of some open covering of Y , say $\mathcal{U}$. But Y is a metric space, and hence paracompact. By Theorem 20.15 ([W], p. 151), every open cover of a paracompact (T_1) space is normal. Hence $\mathcal{U}$ is a normal cover and so there exists a normal sequence of covers $\{\mathcal{U}_i\}$ such that $\mathcal{U}_1$ refines $\mathcal{U}$. The inverse images under f of these covers will be a normal sequence and $f^{-1}(\mathcal{U}_1) = \{f^{-1}(U) \mid U \in \mathcal{U}_1\}$ refines $\mathcal{G}$. Therefore $\mathcal{G}$ is a normal cover of X .

(b) → (c) . Suppose $\mathcal{G}$ is an open cover of X . By assumption, there exists a continuous map $f : X \to Y$, where Y is a metric space, such that $\mathcal{G}$ is refined by the inverse image of some open covering of Y , say $\mathcal{U}$. But every metric space is paracompact. Hence $\mathcal{U}$ has an open locally finite refinement, say $\mathcal{V}$. Consider the inverse image under f of $\mathcal{V}$. Certainly $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open refinement of $\mathcal{G}$ and is locally finite. Finally we can show that $f^{-1}(\mathcal{V})$ is a normal covering as in (b) → (a) .

(c) → (d) . Let $\mathcal{G}$ be an open cover of X . Then by part (c), $\mathcal{G}$ has a refinement $\mathcal{U}$ which is locally finite, open, and is a normal cover. Therefore, there exists a normal sequence of open covers $\{\mathcal{U}_i\}$ such that $\mathcal{U}_1 = \mathcal{U}$ and $\mathcal{U}_{i+1}$ star-refines $\mathcal{U}_i$ for $i = 2,3,\cdots$. Hence, X is pseudometrizable ([W], Theorem 23.4, p. 167). Let ρ be a pseudometric on X . Note that by the same theorem, we may assume that the covers $\mathcal{U}_i$ are open in the ρ - topology.

Consider any $U_\lambda \in \mathcal{U}$, and let $H_\lambda = X - \overline{(X - U_\lambda)}^\rho$, where $\overline{(X - U_\lambda)}^\rho$ denotes the closure of $X - U_\lambda$ in the ρ - topology. Then for each λ , $H_\lambda \subset U_\lambda$, and H_λ is open in the ρ - topology (which is weaker than the original topology). Clearly $\{H_\lambda\}$ is locally finite. We must show that the collection $\{H_\lambda\}$ covers X and that each H_λ is a cozero set. Consider any $x \in X$. Then $x \in U_{2\alpha}$ for some $U_{2\alpha} \in \mathcal{U}_2$. But $U_{2\alpha} \subset U_\gamma$ for some $U_\gamma \in \mathcal{U}$. So we have $x \in U_{2\alpha} \subset U_\gamma$. We will show that $x \in H_\gamma = X - \overline{(X - U_\gamma)}^\rho$. But $x \in X - \overline{(X - U_\gamma)}^\rho$ if and only if $x \notin \overline{X - U_\gamma}^\rho$ if and only if x has a ρ - neighborhood inside U_γ . This is clearly the case since $x \in U_{2\alpha} \subset U_\gamma$, and $U_{2\alpha}$ is open in the ρ - topology. Hence $\{H_\lambda\}$ covers X .

To show that each H_λ is a cozero set, let us define $f_\lambda : X \to [0,1]$ by $f_\lambda(x) = \dfrac{\rho(x,X-H_\lambda)}{1 + \rho(x,X-H_\lambda)}$. Then f_λ is continuous for each λ and furthermore $f_\lambda(H_\lambda) > 0$ and $f_\lambda(X-H_\lambda) = 0$. Hence each H_λ is a cozero set.

(d) $\to$ (e) . Let $f(x) = \sum_\lambda f(x)$ and define $\phi_\lambda(x) = \dfrac{f_\lambda(x)}{f(x)}$. Then ϕ_λ is a continuous function and $0 \leq \phi_\lambda(x) \leq 1$ for every $x \in X$. Consider any $x \in X$. Now x is contained in only finitely many H_λ's , say $H_1,\cdots,H_n$ (since $\{H_\lambda\}$ is locally finite and hence point finite). Therefore $f_\lambda(x) = 0$ and so $\phi_\lambda(x) = 0$ for $\lambda \neq 1,\cdots,n$. Then clearly we have $\sum_\lambda \phi_\lambda(x) = 1$. So $\{\phi_\lambda\}$ is a partition of unity on X .

Now consider any function ϕ_λ in the partition. Since $f_\lambda = 0$ outside H_λ , we have that $\phi_\lambda = \dfrac{f_\lambda}{f} = 0$ outside H_λ . But $H_\lambda \subset G$ for

some element G of the cover $\mathcal{G}$. Hence ϕ_λ vanishes outside some element of the cover $\mathcal{G}$ which implies that the partition of unity $\{\phi_\lambda\}$ is subordinated to $\mathcal{G}$.

(e) $\to$ (f). This is obvious.

(f) $\to$ (a). Let $\{\phi_\lambda \mid \lambda \in \Lambda\}$ be a partition of unity on X which is subordinated to the cover $\mathcal{G}$. Then we have that each ϕ_λ is a continuous map from X into $[0,1]$ such that $\sum_\lambda \phi_\lambda(x) = 1$ for every $x \in X$ and $\{x \mid \phi_\lambda(x) > 0\}$ is contained in some element of the cover for each $\lambda \in \Lambda$. Consider the subset M of the product I^Λ consisting of all points $y \in I^\Lambda$ with $y_\lambda = 0$ except countably often and $\sum_\lambda y_\lambda = 1$. Define a metric d on M by $d(y,z) = \sum_{\lambda\in\Lambda} |y_\lambda - z_\lambda|$. Define a map $\phi : X \to M$ by $[\phi(x)]_\lambda = \phi_\lambda(x)$. Then ϕ is a continuous mapping of X into M. For given a point x_o of X and $\epsilon > 0$, we can find a finite subset Γ of Λ and a neighborhood U_o of x_o such that $\sum_{\lambda\notin\Gamma} \phi_\lambda(x_o) < \epsilon$ and $\sum_{\lambda\in\Gamma} |\phi_\lambda(x) - \phi_\lambda(x_o)| < \epsilon$ for $x \in U_o$. But then

$$\sum_{\lambda\notin\Gamma} \phi_\lambda(x) = \sum_{\lambda\in\Gamma} (\phi_\lambda(x_o) - \phi_\lambda(x)) + \sum_{\lambda\notin\Gamma} \phi_\lambda(x_o)$$

$$\leq \sum_{\lambda\in\Gamma} |\phi_\lambda(x) - \phi_\lambda(x_o)| + \sum_{\lambda\notin\Gamma} \phi_\lambda(x_o) \leq 2\epsilon .$$

Hence

$$d(\phi(x),\phi(x_o)) < \sum_{\lambda\notin\Gamma} |\phi_\lambda(x) - \phi_\lambda(x_o)| + \epsilon \leq \sum_{\lambda\notin\Gamma} (\phi_\lambda(x) + \phi_\lambda(x_o))$$

$$+ \epsilon < 2\epsilon + \epsilon + \epsilon = 4\epsilon$$

for every $x \in U_o$.

Let $V_\lambda = \{y \in M \mid y_\lambda > 0\}$. Then the V_λ's are open subsets of M and $\phi(X) \subset \bigcup_\lambda V_\lambda$; that is, $\{V_\lambda\}$ is an open cover of $\phi(X)$. Furthermore, $\phi^{-1}(V_\lambda) = \{x \mid \phi_\lambda(x) > 0\}$. So $\phi^{-1}(V_\lambda)$ is contained in some member of G . Hence by (b) , G is a normal covering.

(a) $\to$ (g) . Let G be a normal open cover of X . Then there exists a normal sequence of open covers $\{G_i\}$ such that G_1 refines G and G_{i+1} star-refines G_i for every $i = 1,2,\cdots$. Clearly every open cover in a normal sequence is a normal cover. Consider $G_1 = \{U_\gamma\}$, and let us look at any one of the subspaces U_γ . We must show that $\{G \cap U_\gamma \mid G \in G\}$ is a normal cover of U_γ . Let $H_i = \{G_i \cap U_\gamma \mid G_i \in G_i\}$ for every $i > 1$. Then clearly $\{H_i\}$ is a normal sequence of open covers of U_γ , and H_2 refines $\{G \cap U_\gamma \mid G \in G\}$. Hence there exists a normal open covering $\{U_\gamma\}$ of X such that on each of the subspaces U_γ , the covering $\{G \cap U_\gamma \mid G \in G\}$ is normal.

(g) $\to$ (a) . Let $\{V_\lambda\}$ be a normal open covering of X which is a star-refinement of $\{U_\gamma\}$. Let $\{\phi_\lambda \mid \lambda \in \Lambda\}$ be a partition of unity on X which is subordinated to $\{V_\lambda\}$. We will assume that $\{x \mid \phi_\lambda(x) > 0\} \subset V_\lambda$ for each $\lambda \in \Lambda$. Now $\overline{V}_\lambda$ is contained in some element U_γ of $\{U_\gamma\}$ and so $\{G \cap \overline{V}_\lambda \mid G \in G\}$ is a normal covering of the subspaces $\overline{V}_\lambda$. Let $\{f_{\lambda_\mu} \mid \mu \in \Gamma_\lambda\}$ be a parition of unity on $\overline{V}_\lambda$ which is subordinated to this covering. Put

$$\phi_{\lambda_\mu}(x) = \begin{cases} \phi_\lambda(x) \cdot f_{\lambda_\mu}(x) & \text{for} \quad x \in \overline{V}_\lambda \\ 0 & \text{for} \quad x \in X - \overline{V}_\lambda \end{cases}$$

Now $\phi_\lambda(x) = 0$ for $x \in \overline{V}_\lambda - V_\lambda$. Therefore $\phi_{\lambda_\mu}(x)$ is continuous over X. Since $\phi_\lambda(x) = \sum_{\mu\in\Gamma_\lambda} \phi_{\lambda_\mu}(x)$ for $x \in X$, we have $\sum_{\lambda,\mu} \phi_{\lambda_\mu}(x) = 1$ for $x \in X$. Moreover, we have that

$$\{x \mid \phi_{\lambda_\mu}(x) > 0\} \subset \{x \mid \phi_\lambda(x) > 0\} \cap \{x \mid f_{\lambda_\mu}(x) > 0\} \quad ,$$

and that $\{x \mid f_{\lambda_\mu}(x) > 0\}$ is contained in some element of G. Hence $\{\phi_{\lambda_\mu} \mid \lambda \in \Lambda , \mu \in \Gamma_\lambda\}$ is a parition of unity on X which is subordinated to G. Hence, by (f), G is normal.

The remaining two theorems in this section are vital to the proofs of the two main lemmas which will be needed to establish the major result. Each is a consequence of Theorem 2.8. The first gives an important characterization of countable paracompactness and normality in terms of normal coverings. Theorem 2.10 provides a sufficient condition that guarantees that a given cover of a topological space is normal.

Theorem 2.9. Let X be a topological space. Then X is countably paracompact and normal if and only if every countable open covering of X is a normal covering[9].

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9. Morita states and proves a more general form of this result. He is able to prove that X is m - paracompact and normal if and only if every open covering of X with power $\leq m$ is a normal covering. (X is said to be m - PARACOMPACT if every open cover of power m has an open locally finite refinement).

Proof: $\longrightarrow$. Suppose X is countably paracompact and normal, and let $G = \{G_n\}$ be a countable open covering of X . Since X is countably paracompact, $\{G_n\}$ is shrinkable; that is, it has an open refinement $\{V_n\}$ with $\overline{V}_n \subset G_n$ for each n ([W], p. 156). Then for each n , $\overline{V}_n$ and $X - G_n$ are disjoint closed sets in X . Hence by normality, there exists a Urysohn function $f_n : X \to [0,1]$ such that $f_n(\overline{V}_n) = 1$ and $f_n(X - G_n) = 0$. Consider $\phi_n = \frac{f_n}{f}$ where $f = \sum_n f_n$. We will show that $\{\phi_n\}$ is a partition of unity subordinated to the cover $\{G_n\}$. Clearly, each ϕ_n is continuous since f_n is continuous for each n (and f is thus continuous). Consider any $x \in X$. We have that

$$\sum_n \phi_n(x) = \frac{f_1(x) + f_2(x) + \cdots}{f(x)} = 1 \quad .$$

Furthermore, each ϕ_n vanishes outside some element of the cover $\{G_n\}$ since for each k , the function ϕ_k vanishes outside G_k . Therefore, $\{\phi_n\}$ is a partition of unity subordinated to the cover $\{G_n\}$. So, by Theorem 2.8, $\{G_n\}$ is a normal covering.

$\longleftarrow$. Suppose every countable open cover of X is a normal cover. By part (c) of Theorem 2.8, we have that every countable open cover has an open locally finite refinement. So X is countably paracompacy.

To show that X is normal, let G and H be any two disjoint closed subsets of X . Then $\{X - G , X - H\}$ is a countable open cover of X , and hence is normal. By part (f) of Theorem 2.8, there exists a partition of unity $\{\phi_\alpha\}$ subordinated to the cover $\{X - G , X - H\}$. Now each element of the partition must vanish outside some element of the cover.

So there exists a function $\phi_{\alpha_1} \in \{\phi_\alpha\}$ such that ϕ_{α_1} vanishes outside $X - H$; that is, $\phi_{\alpha_1}(H) = 0$. Similarly, there exists a function $\phi_{\alpha_2} \in \{\phi_\alpha\}$ such that ϕ_{α_2} vanishes outside $X - G$; that is $\phi_{\alpha_2}(g) = 0$. Consider $f = \dfrac{\phi_{\alpha_1}}{\phi_{\alpha_1} + \phi_{\alpha_2}}$. Then f is a continuous function from X into $[0,1]$ and $f(G) = 1$, $f(H) = 0$. Hence X is normal by Urysohn's Lemma.

<u>Theorem 2.10</u>. Let $\mathcal{G} = \sum_{i=1}^{\infty} \mathcal{G}_i$ be a σ - locally finite open covering of a topological space X where each $\mathcal{G}_i = \{G_{i\alpha} \mid \alpha \in \Omega_i\}$ is a locally finite collection of subsets of X . If for each $G_{i\alpha}$ there exists a continuous mapping $\psi_{i\alpha} : X \to [0,1]$ such that $G_{i\alpha} = \{x \mid \psi_{i\alpha}(x) > 0\}$, then $\mathcal{G}$ is a normal covering.

<u>Proof</u>: Each $\mathcal{G}_i$ is locally finite, so $\psi_i(x) = \sum_{\alpha\in\Omega_i} \psi_{i\alpha}(x)$ is a continuous function over X . For $x \in X$, let us put

$$\psi(x) = \sum_{i=1}^{\infty} \frac{\psi_i(x)}{2^i(1 + \psi_i(x))} .$$

Then ψ is a continuous function over X , and since $\mathcal{G}$ is a covering of X , we have that $\psi(x) > 0$ for every $x \in X$. Now for $x \in X$, let $\Psi_{i\alpha}(x) = \dfrac{\psi_{i\alpha}(x)}{2^i\psi(x)(1 + \psi_i(x))}$. Then we have that $\sum_{i=1}^{\infty} \sum_{\alpha\in\Omega_i} \Psi_{i\alpha}(x) = 1$ for $x \in X$, and $G_{i\alpha} = \{x \mid \Psi_{i\alpha}(x) > 0\}$. Hence $\{\Psi_{i\alpha} \mid \alpha \in \Omega_i , i = 1,2,\cdots\}$ is a partition of unity subordinated to $\mathcal{G}$. So by Theorem 2.8, $\mathcal{G}$ is a normal covering.

2.4. Baire Spaces. Let Ω be a non-empty set, and let $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ be any two sequences of elements from Ω. If we define $\rho(\alpha,\beta)$ by

$$\rho(\alpha,\beta) = \begin{cases} \frac{1}{k} & \text{if } \alpha_i = \beta_i \quad \text{for } i < k \text{ and } \alpha_k \neq \beta_k \\ 0 & \text{if } \alpha_i = \beta_i \quad \text{for } i = 1,2,\cdots \end{cases}$$

then the set of all sequences of elements from Ω determine a metric space. This metric space is denoted by $N(\Omega)$ and is called a BAIRE SPACE. If we put $V(\alpha_1,\cdots,\alpha_i) = \{(\beta_j) \mid \beta_1 = \alpha_1,\cdots,\beta_i = \alpha_i \; ; (\beta_j) \in N(\Omega)\}$, then the family $\{V(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \; ; \; i = 1,2,\cdots\}$ is an open basis of $N(\Omega)$. Hence the weight[10] of $N(\Omega)$ is equal to $|\Omega|$ or χ_o according as $|\Omega| \geq \chi_o$ or $2 \leq |\Omega| \leq \chi_o$.

The importance of Baire spaces lies in the fact that the two main lemmas preceeding our major result provide a characterization of a normal P - space in terms of a normal product with a subspace S of the Baire space $N(\Omega)$. The main result then follows since, for any metric space, we can find a subspace of the Baire space and a closed continuous mapping from this subspace onto the metric space. Essentially this is the statement of Theorem 2.11 which follows immediately. It should be pointed out that Theorem 2.11 is an abbreviated form of a more general result which Morita proves.

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10. The WEIGHT of a space is the least cardinal of an open basis.

Theorem 2.11. Let Ω be a set of power m where $m \geq \aleph_0$. If X is a metric space of weight $\leq m$, then there exists a subspace S of the Baire space $N(\Omega)$ and a closed continuous map g from S onto X such that $g^{-1}(x)$ is compact for every $x \in X$.

Proof: Let X be a metric space of weight $\leq m$. Then there exists a countable family $\{G_i\}$ of open coverings of X such that each G_i is a locally finite covering consisting of subsets of diameter $\leq 2^{-i}$ with respect to some metric on X. (The existence of such a family is guaranteed by a well-known theorem of A.H. Stone.) By assumption, we may index the sets of G_i using elements of Ω, so let $G_i = \{G(\alpha;i) \mid \alpha \in \Omega\}$. Now let S be the set of points $\alpha = (\alpha_1, \alpha_2, \cdots)$ of $N(\Omega)$ such that $\bigcap_{i=1}^{\infty} \overline{G(\alpha_i;i)} \neq \phi$, and put $g(\alpha) = \bigcap_{i=1}^{\infty} \overline{G(\alpha_i;i)}$ for $(\alpha_1, \alpha_2, \cdots) \in S$. Then $g(\alpha)$ consists of a single point and g is a closed continuous map from S onto X. Furthermore, $g^{-1}(x)$ is compact for each point $x \in X$. (For a proof of these statements, see Mo_1.)

The following theorem provides a condition which is sufficient to guarantee that a zero dimensional space is homeomorphic to some subset of the Baire space $N(\Omega)$. The dimension of a space can be defined several ways. The definition we will use is: a space is said to have DIMENSION $\leq n$ if for any finite open covering of the space, there exists an open refinement of order[11] $\leq n+1$. Therefore, X is said to have dimension zero if every finite open covering of X can be refined by a cover of disjoint open sets.

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11. A covering of a space is said to have ORDER n if each point of the space belongs to at most n elements of the cover.

Theorem 2.12. Let X be a space of dimension zero. If Ω is a set whose cardinal number is not less than the cardinal number of a basis of open sets of X, then X is homeomorphic to a subset of $N(\Omega)$.

Proof: Let X be a space of dimension zero. Then there exists a countable collection of open coverings $\{\mathcal{U}_n\}$ of X such that
$\mathcal{U} = \{U \mid U \in \mathcal{U}_i , i = 1,2,\cdots\}$ is a basis of open sets of X and
$\mathcal{U}_n = \{U(\alpha_1,\cdots,\alpha_n) | \alpha_i \in \Omega , i=1,2,\cdots,n\}$, $U(\alpha_1,\cdots,\alpha_n) \cap U(\beta_1,\cdots,\beta_n) = 0$ for $(\alpha_1,\cdots,\alpha_n) \neq (\beta_1,\cdots,\beta_n)$, and $U(\alpha_1,\cdots,\alpha_{n-1}) = \bigcup_{\beta\in\Omega} U(\alpha_1,\cdots,\alpha_{n-1},\beta)$.
Now for every point $p \in X$ and any n, there exists only one non-empty set $U(\alpha_1,\cdots,\alpha_n)$ of $\mathcal{U}_n$ containing p. Hence define a mapping of X into $N(\Omega)$ by $f(p) = (\alpha_1,\alpha_2,\cdots)$ where $p \in U(\alpha_1,\cdots,\alpha_n)$ for $n=1,2,\cdots$. It is easily verified that f is a topological mapping of X onto a subset of $N(\Omega)$.

2.5. Preliminary Lemmas. In this section we will state and prove the four principal results from which our main theorem follows. The first result is stated as a theorem since it is very interesting in its own right. It is well known that every closed subset of a normal space is normal. The theorem gives a condition under which an open subset of a normal space is normal, namely if that subset is an F_σ.

Theorem 2.13. An open F_σ subset of a normal space is normal.

Proof: Let X be a normal space and let U be an open F_σ subset of X. Then by normality, U can be written as $U = \bigcup_{n=1}^{\infty} F_n$ where F_n is closed (and hence normal) in X and $F_n \subset \text{Int. } F_{n+1}$ for all n.

Let A be closed in U and let $f : A \to [0,1]$ be continuous. Furthermore, let $f_n = f \mid (F_n \cap A)$. By the normality of F_1, f_1 can be extended to a continuous map $g_1 : F_1 \to [0,1]$. Now by the normality of F_2, the map $g_1 \cup f_2$ can be extended from $(F_1 \cup A) \cap F_2$ to a map $g_2 : F_2 \to [0,1]$. Continuing inductively, we can define a continuous map $g_n : F_n \to [0,1]$ for each n such that $g_n \mid F_{n-1} = g_{n-1}$. The maps $g(x) = g_n(x)$, $x \in F_n$, is then an extension of f, and is continuous at every $x \in U$ since $x \in \text{Int. } F_n$ for some n (so that g coincides with some g_n of a neighborhood of x). Hence U is normal.

Corollary. An open F_σ subset of a countably paracompact and normal space is countably paracompact and normal.

Proof: Let X be countably paracompact and normal, and let $U = \bigcup_{n=1}^{\infty} F_n$ be an open F_σ subset of X. Consider $U \times I = \bigcup_{n=1}^{\infty} (F_n \times I)$. Then by the argument above, $U \times I$ is normal. But I is a compact metric space, so U is countably paracompact and normal ([W], Theorem 21.4, p. 157).

The major result of this section follows immediately from the following lemmas. Note that Lemma 2.15 is the converse of Lemma 2.14.

Lemma 2.14. Let X be a normal $P(m)$ - space and Ω a set of power m, where $m \geq 2$. If S is a subspace of the Baire space $N(\Omega)$, then the product space $X \times S$ is normal.

Proof: Let $\mathcal{M} = \{M_i \mid i = 1,2,\cdots\}$ be any countable open covering of $X \times S$, where S is a subspace of $N(\Omega)$. Let

$$V(\alpha_1,\cdots,\alpha_i) = \{(\beta_j) \mid \beta_1 = \alpha_1,\cdots,\beta_i = \alpha_i \; ; \; (\beta_j) \in N(\Omega)\} \quad .$$

Then $\{V(\alpha_1,\cdots,\alpha_i) \cap S \mid \alpha_1,\cdots,\alpha_i \in \Omega \; ; \; i = 1,2,\cdots\}$ forms an open basis of S . For a refinement of M , we can find an open cover

$$(1) \quad \{L(\alpha_1,\cdots,\alpha_i;k)\times[V(\alpha_1,\cdots,\alpha_i)\cap S] \mid \alpha_1,\cdots,\alpha_i\in\Omega \; ; \; i=1,2,\cdots \; ; \; k=1,2,\cdots\}$$

such that the $L(\alpha_1,\cdots,\alpha_i;k)$ are open subsets of X and

$$(2) \qquad L(\alpha_1,\cdots,\alpha_i;k) \times [V(\alpha_1,\cdots,\alpha_i) \cap S] \subset M_k$$

$$\text{for } \alpha_1,\cdots,\alpha_i \in \Omega \; ; \; i = 1,2,\cdots \; ; \; k = 1,2,\cdots \; .$$

Note that $L(\alpha_1,\cdots,\alpha_i;k) = \phi$ if $V(\alpha_1,\cdots,\alpha_i) \cap S = \phi$.

Now for $j \leq i$, we have that

$$L(\alpha_1,\cdots,\alpha_j;k) \times [V(\alpha_1,\cdots,\alpha_j,\cdots,\alpha_i) \cap S] \subset L(\alpha_1,\cdots,\alpha_j;k) \times [V(\alpha_1,\cdots,\alpha_j) \cap S] \subset M_k \quad .$$

Let us put $G(\alpha_1,\cdots,\alpha_i;k) = \bigcup_{j=1}^{i} L(\alpha_1,\cdots,\alpha_j;k)$. Then we have that

$$(3) \qquad G(\alpha_1,\cdots,\alpha_i;k) \times [V(\alpha_1,\cdots,\alpha_i) \cap S] \subset M_k \quad , \quad \text{and}$$

$$(4) \qquad G(\alpha_1,\cdots,\alpha_i;k) \subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1};k) \quad .$$

Now let

$$(5) \qquad G(\alpha_1,\cdots,\alpha_i) = \bigcup_{k=1}^{\infty} G(\alpha_1,\cdots,\alpha_i;k) \quad .$$

Obviously from (4) we have that

$$(6) \qquad G(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1}) \quad \text{for} \quad \alpha_1,\cdots,\alpha_{i+1} \in \Omega \ .$$

We must show that

$$(7) \qquad X = \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i) \quad \text{for} \quad (\alpha_1,\alpha_2,\cdots) \in S \ .$$

To prove this, let $(\alpha_1,\alpha_2,\cdots)$ be any point of S . Now $(\alpha_1,\alpha_2,\cdots) \in V(\beta_1,\cdots,\beta_i) \cap S$ if and only if $\beta_1 = \alpha_1,\cdots,\beta_i = \alpha_i$. Also (1) is a covering of $X \times S$. Then from (5) we get that

$$X \times (\alpha_1,\alpha_2,\cdots) \subset \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i) \times [V(\alpha_1,\cdots,\alpha_i) \cap S] \ .$$

Hence (7) follows.

By assumption, X is a normal $P(m)$ - space. Therefore, by Lemma 2.1, there exists a family $\{H(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega \ ; i=1,2,\cdots\}$ of open F_σ subsets of X such that

$$(8) \qquad H(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i) \quad \text{for} \quad \alpha_1,\cdots,\alpha_i \in \Omega \ , \quad \text{and}$$

$$(9) \qquad X = \bigcup_{i=1}^{\infty} H(\alpha_1,\cdots,\alpha_i) \quad \text{for} \quad (\alpha_1,\alpha_2,\cdots) \in S \ .$$

By Theorem 2.4, we have that X is countably paracompact and normal. Now each $H(\alpha_1,\cdots,\alpha_i)$ is an open F_σ subset of X , and, by the Corollary to Theorem 2.13, each $H(\alpha_1,\cdots,\alpha_i)$ is countably paracompact and normal. Consider the countable open covering

$$\{G(\alpha_1,\cdots,\alpha_i;k) \cap H(\alpha_1,\cdots,\alpha_i) \mid k = 1,2,\cdots\}$$

of the subspaces $H(\alpha_1,\cdots,\alpha_i)$. By Lemma 2.7, it is a normal covering. Hence, by Lemma 2.6, the cover

$$\{[G(\alpha_1,\cdots,\alpha_i;k) \cap H(\alpha_1,\cdots,\alpha_i)] \times [V(\alpha_1,\cdots,\alpha_i)\cap S] \mid k = 1,2,\cdots\}$$

is a normal covering of the subspace $H(\alpha_1,\cdots,\alpha_i) \times [V(\alpha_1,\cdots,\alpha_i)\cap S]$ of $X\times S$.

Now X is normal and each $H(\alpha_1,\cdots,\alpha_i)$ is an open F_σ subset of X . Thus, there exists a continuous mapping $\psi(\ ;\alpha_1,\cdots,\alpha_i) : X \to [0,1]$ such that $H(\alpha_1,\cdots,\alpha_i) = \{x \mid \psi(x;\alpha_1,\cdots,\alpha_i) > 0\}$. Likewise for each $V(\alpha_1,\cdots,\alpha_i) \cap S$, there exists a continuous map $\Psi(\ ;\alpha_1,\cdots,\alpha_i) : S \to [0,1]$ such that

$$V(\alpha_1,\cdots,\alpha_i) \cap S = \{s \mid \Psi(s;\alpha_1,\cdots,\alpha_i) > 0\} .$$

So

$$(10) \qquad H(\alpha_1,\cdots,\alpha_i) \times [V(\alpha_1,\cdots,\alpha_i) \cap S]$$

$$= \{(x,s) \mid \psi(x;\alpha_1,\cdots,\alpha_i)\Psi(s,\alpha_1,\cdots,\alpha_i) > 0\}$$

and $\psi(x;\alpha_1,\cdots,\alpha_i)\ \Psi(s;\alpha_1,\cdots,\alpha_i)$ is continuous over $X\times S$. But $\{V(\alpha_1,\cdots,\alpha_i) \cap S \mid \alpha_1,\cdots,\alpha_i \in \Omega\ ;\ i = 1,2,\cdots\}$ is a σ - locally finite collection in S . Therefore the family

$$\mathcal{R} = \{H(\alpha_1,\cdots,\alpha_i) \times [V(\alpha_1,\cdots,\alpha_i)\cap S] \mid \alpha_1,\cdots,\alpha_i \in \Omega\ ;\ i = 1,2,\cdots\}$$

is σ - locally finite in $X \times S$, and from (9), $\mathcal{R}$ covers $X \times S$. So the family $\mathcal{R}$ of subsets of $X \times S$ is a σ - locally finite open covering of $X \times S$, and by Theorem 2.10, it is a normal covering of $X \times S$.

We have already shown that $\{[G(\alpha_1,\cdots,\alpha_i;k) \cap H(\alpha_1,\cdots,\alpha_i)] \times [V(\alpha_1,\cdots,\alpha_i) \cap S] \mid k = 1,2,\cdots\}$ is a normal cover of the subspace $H(\alpha_1,\cdots,\alpha_i) \times [V(\alpha_1,\cdots,\alpha_i) \cap S]$ and moreover, it is a refinement of the covering $\{H(\alpha_1,\cdots,\alpha_i) \times [V(\alpha_1,\cdots,\alpha_i) \cap S] \cap M_k \mid k = 1,2,\cdots\}$ by virtue of (3). Therefore, by part (g) of Theorem 2.8, $\mathcal{M}$ is a normal covering of $X \times S$. But $\mathcal{M}$ is an arbitrary countable open covering of $X \times S$. Hence by Theorem 2.9, $X \times S$ is countably paracompact and normal.

Lemma 2.15. Let X be a topological space and Ω a set of power m, where $m \geq 2$. If the product space $X \times S$ is normal for every subspace S of the Baire space $N(\Omega)$, then X is a normal $P(m)$ - space.

Proof: First, we will show that $X \times S$ is countably paracompact for every subspace S of the Baire space $N(\Omega)$. Let C be the Cantor discontinuum; that is $C = D^{\chi_0}$ where D is a discrete space of two points. Then $S \times C$ is zero-dimensional in the covering sense. Also, the weight is not greater than m (for the case where $2 \leq |\Omega| \leq \chi_0$) or χ_0 (for the case where $|\Omega| \geq \chi_0$). Hence the product space $S \times C$ is homeomorphic to a subspace $N(\Omega)$ by Theorem 2.12. By assumption, $X \times S$ is normal for every subspace S of $N(\Omega)$. Therefore, $X \times (S \times C)$ is normal and so $(X \times S) \times C$ is normal (since $X \times (S \times C)$ is homeomorphic to $(X \times S) \times C$). Since C is a compact metric space, it follows that $X \times S$ is countably paracompact ([W], Theorem 21.4, p. 157).

Now let $\{G(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega ; i = 1,2,\cdots\}$ be a family of open subsets of X such that $G(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$ for $\alpha_1,\cdots,\alpha_{i+1} \in \Omega$. Let us put

$$(1) \qquad S = \{(\alpha_1,\alpha_2,\cdots) \mid \bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i) = X\} .$$

Then S is a subspace of $N(\Omega)$. Consider the family of sets

$$V(\alpha_1,\cdots,\alpha_i) = \{(\beta_1,\beta_2,\cdots) \mid \beta_j = \alpha_j \text{ for } j \leq i ; (\beta_j) \in N(\Omega)\} .$$

Then the family

$$(2) \quad \{G(\alpha_1,\cdots,\alpha_i)\times[V(\alpha_1,\cdots,\alpha_i)\cap S] \mid \alpha_1,\cdots,\alpha_i \in \Omega ; i = 1,2,\cdots\}$$

is an open covering of $X\times S$. Now $\{V(\alpha_1,\cdots,\alpha_i) \cap S \mid \alpha_1,\cdots,\alpha_i \in \Omega\}$ is locally finite in S. Therefore the covering (2) is a σ - locally finite open covering of $X\times S$, and since the space $X\times S$ is countably paracompact and normal, it follows that the covering (2) is a normal covering by Lemma 2.7.

Let $\{L_\lambda \mid \lambda \in \Lambda\}$ be a locally finite open covering of $X\times S$ such that $\{\overline{L}_\lambda \mid \lambda \in \Lambda\}$ is a refinement of the covering (2). We can find an open refinement of $\{L_\lambda \mid \lambda \in \Lambda\}$ of the form

$$(3) \quad \{L(\alpha_1,\cdots,\alpha_i;\lambda)\times[V(\alpha_1,\cdots,\alpha_i)\cap S] \mid \alpha_1,\cdots,\alpha_i \in \Omega , \lambda \in \Lambda ; i=1,2,\cdots\}$$

where

$$(4) \qquad L(\alpha_1,\cdots,\alpha_i;\lambda) \times [V(\alpha_1,\cdots,\alpha_i)\cap S] \subset L_\lambda ,$$

and where we assume $L(\alpha_1,\cdots,\alpha_i;\lambda) = \phi$ if $V(\alpha_1,\cdots,\alpha_i) \cap S = \phi$.

Now, for each $\lambda \in \Omega$, there exists a finite sequence $(\beta_1,\cdots,\beta_j)$ of elements of Ω such that

(5) $$\overline{L}_\lambda \subset G(\beta_1,\cdots,\beta_j) \times [V(\beta_1,\cdots,\beta_j) \cap S] \quad .$$

Suppose that $L(\alpha_1,\cdots,\alpha_i;\lambda) \neq \phi$, $V(\alpha_1,\cdots,\alpha_i) \cap S \neq \phi$. Then we have $\overline{L(\alpha_1,\cdots,\alpha_i;\lambda)} \times [V(\alpha_1,\cdots,\alpha_i) \cap S] \subset \overline{L}_\lambda \subset G(\beta_1,\cdots,\beta_j) \times [V(\alpha_1,\cdots,\alpha_j) \cap S]$. Two cases arise:

(a) If $j \leq i$, then we have $\beta_1 = \alpha_1,\cdots,\beta_j = \alpha_j$, and $G(\beta_1,\cdots,\beta_j) = G(\alpha_1,\cdots,\alpha_j) \subset G(\alpha_1,\cdots,\alpha_j,\cdots,\alpha_i)$. Therefore, $\overline{L(\alpha_1,\cdots,\alpha_i;\lambda)} \times [V(\alpha_1,\cdots,\alpha_i) \cap S] \subset \overline{L}_\lambda \subset G(\alpha_1,\cdots,\alpha_i) \times [V(\alpha_1,\cdots,\alpha_i) \cap S]$.

(b) If $j > i$, then $\beta_1 = \alpha_1,\cdots,\beta_i = \alpha_i$, and $V(\alpha_1,\cdots,\alpha_i) \cap S = V(\alpha_1,\cdots,\alpha_i,\beta_{i+1},\cdots,\beta_j) \cap S$. Then

$$\overline{L(\alpha_1,\cdots,\alpha_i;\lambda)} \times [V(\alpha_1,\cdots,\alpha_i,\beta_{i+1},\cdots,\beta_j) \cap S]$$
$$\subset G(\alpha_1,\cdots,\alpha_i,\beta_{i+1},\cdots,\beta_j) \times [V(\alpha_1,\cdots,\alpha_i,\beta_{i+1},\cdots,\beta_j) \cap S] \ .$$

Now for $\alpha_1,\cdots,\alpha_i \in \Omega$ and $\lambda \in \Lambda$, let $M(\alpha_1,\cdots,\alpha_i;\lambda)$ be the union of all those sets $L(\alpha_1,\cdots,\alpha_j;\lambda)$ with $j \leq i$ which satisfy the following two conditions:

(6) $$V(\alpha_1,\cdots,\alpha_j) \cap S = V(\alpha_1,\cdots,\alpha_j,\cdots,\alpha_i) \cap S \quad ,$$

(7) $$\overline{L(\alpha_1,\cdots,\alpha_j;\lambda)} \subset G(\alpha_1,\cdots,\alpha_j,\cdots,\alpha_i) \quad .$$

Note that the above shows that for each $L(\alpha_1,\cdots,\alpha_j;\lambda)$, there is some $L(\alpha_1,\cdots,\alpha_i;\lambda)$ with $j \leq i$ satisfying (6) and (7). Hence the collection

$$(8) \quad \{M(\alpha_1,\cdots,\alpha_i,\lambda)\times[V(\alpha_1,\cdots,\alpha_i)\cap S] \mid \alpha_1,\cdots,\alpha_i\in\Omega \ , \ \lambda\in\Lambda \ ; \ i=1,2,\cdots\}$$

is an open covering of $X\times S$, and

$$(9) \qquad M(\alpha_1,\cdots,\alpha_i;\lambda) \times [V(\alpha_1,\cdots,\alpha_i) \cap S] \subset L_\lambda \quad ,$$

$$(10) \qquad \overline{M(\alpha_1,\cdots,\alpha_i;\lambda)} \subset G(\alpha_1,\cdots,\alpha_i) \quad .$$

Note that $M(\alpha_1,\cdots,\alpha_i;\lambda)$ is a sum of a finite number of subsets $L(\alpha_1,\cdots,\alpha_j;\lambda)$, $j = 1,2,\cdots,i$, which satisfy (4), (6), and (7), and so (9) and (10) hold.

Finally, let us put

$$(11) \quad F(\alpha_1,\cdots,\alpha_i) = \begin{cases} \cup\{\overline{M(\alpha_1,\cdots,\alpha_i;\lambda)} \mid \lambda\in\Lambda\} & \text{for} \quad V(\alpha_1,\cdots,\alpha_i) \cap S \neq \phi \\ \\ \phi & \text{for} \quad V(\alpha_1,\cdots,\alpha_i) \cap S \neq \phi \end{cases}$$

When $V(\alpha_1,\cdots,\alpha_i) \cap S \neq \phi$, since $\{L_\lambda\}$ is locally finite and by (a), it follows that $F(\alpha_1,\cdots,\alpha_i)$ is a closed subset of X . From (10), we know that

$$(12) \qquad F(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i) \quad .$$

Since the family (8) is a covering of $X\times S$, the family

$$\{F(\alpha_1,\cdots,\alpha_i) \times [V(\alpha_1,\cdots,\alpha_i)\cap S] \mid \alpha_1,\cdots,\alpha_i\in\Omega \ ; \ i=1,2,\cdots\}$$

is also a covering of $X\times S$. Hence $X = \bigcup_{i=1}^{\infty} F(\alpha_1,\cdots,\alpha_i)$ for $(\alpha_1,\alpha_2,\cdots\cdot) \in S$. Therefore X is a $P(m)$ - space. Since X is obviously normal, we conclude that X is a normal $P(m)$ - space.

Lemmas 2.14 and 2.15 provide the major vehicles through which we will prove our major result. The following lemma will also be needed.

<u>Lemma 2.16</u>. Let f_i be a closed continuous map of a topological space X_i onto another topological space Y_i such that $f_i^{-1}(y)$ is compact for every point $y \in Y_i$, $i = 1,2,$. If we put $g(x_1,x_2) = (f_1(x_1),f_2(x_2))$ for $x_1 \in X_1$, $x_2 \in X_2$, then g is a closed continuous map of $X_1\times X_2$ onto $Y_1\times Y_2$.

<u>Proof</u>: Let A be any closed subset of $X_1\times X_2$ and suppose that $(y_1,y_2) \in \overline{g(A)}$. Then for any open set H_i of Y_i such that $y_i \in H_i$, we have $(H_1\times H_2) \cap g(A) \neq \phi$. Hence $(f_1^{-1}(H_1) \times f_2^{-1}(H_2)) \cap A \neq \phi$, and so $(f_1^{-1}(y_1) \times f_2^{-1}(y_2)) \cap A \neq \phi$. If not, there would exist an open set G_1 of X_1 and an open set G_2 of X_2 such that $(G_1\times G_2) \cap A = \phi$ with $f_i^{-1}(y_i) \subset G_i$ for $i = 1,2,$ since $f_i^{-1}(y_i)$ is compact for $i = 1,2,\cdots$. Then we would have $(f_1^{-1}(L_1) \times f_2^{-1}(L_2)) \cap A = \phi$ where $L_i = Y_i - f_i(X_i - G_i)$, $i = 1,2,$ because $f_i^{-1}(L_i) \subset G_i$ and f_i is closed (for if not, $L_1\times L_2$ would be an open set containing (y_1,y_2) and missing $\overline{g(A)}$). Hence $(y_1,y_2) \in g(A)$ and so g is a closed mapping.

2.6. Main Theorems. We can now state and prove Morita's results for normal products.

Theorem 2.17. Let X be a topological space and let $m \geq \aleph_0$. Then $X \times Y$ is normal for every metric space Y of weight $\leq m$ if and only if X is a normal $P(m)$ - space.

Proof: $\longrightarrow$. Let $X \times Y$ be normal for every metric space Y of weight $\leq m$. Consider any subspace S of the Baire space $N(\Omega)$. Then S is a metric space of weight $\leq m$, and by assumption $X \times S$ is normal. Hence, by Lemma 2.15, X is a normal $P(m)$ - space.

$\longleftarrow$. Let X be a normal $P(m)$ - space and Y any metric space of weight $\leq m$. Then, by Theorem 2.11, there exists a subspace S of the Baire space $N(\Omega)$ and a closed continuous map g from S onto Y such that $g^{-1}(y)$ is compact for every $y \in Y$. Now define $f : X \times S \to X \times Y$ by $f(x,s) = (x,g(s))$ for $x \in X$, $s \in S$. Then by Lemma 2.16, f is closed and continuous. However, by Lemma 2.14, $X \times S$ is normal. Therefore $X \times Y$ is normal since the closed and continuous image of a normal space is normal.

The following theorem is an immediate consequence of Theorem 2.17.

Theorem 2.18. Let X be a topological space. Then the product space $X \times Y$ is normal for every metric space Y if and only if X is a normal P - space.

Theorem 2.18 represents Morita's answer to the question: what spaces X have a normal product with every metric space Y? Using this result, we have been able to prove a similar result for paracompact products.

Theorem 2.19. Let X be a topological space. Then the product space $X \times Y$ is paracompact for every metric space Y if and only if X is a paracompact P - space.

Proof: $\longrightarrow$. Suppose that $X \times Y$ is paracompact for every metric space Y . Clearly X must be paracompact. Since every paracompact space is normal, it follows that $X \times Y$ is normal for every metric space Y . Then, by Theorem 2.18, X must be a normal P - space. Hence, X is a paracompact P - space.

$\longleftarrow$. Let X be a paracompact P - space, and let Y be any metric space. Consider the Stone-Cech compactification (§5.2), $\beta(X \times Y)$, of the product space $X \times Y$. $\beta(X\ Y)$ is a compact Hausdorff space and so $X \times \beta(X \times Y)$ is paracompact and hence normal. From the Corollary to Theorem 2.5, it follws that $X \times \beta(X \times Y)$ is a P - space. So $X \times \beta(X \times Y)$ is a normal P - space and, by Theorem 2.18, it follows that $[X \times \beta(X \times Y)] \times Y$ is normal for every metric space Y . But $[X \times \beta(X \times Y)] \times Y$ is homeomorphic to $(X \times Y) \times \beta(X \times Y)$, and so $(X \times Y) \times \beta(X \times Y)$ is normal. Hence, by Tamano's theorem (§5.2), $X \times Y$ is paracompact.

§3. Paracompact Spaces.

3.1. Introduction. In this section we will consider the product $X \times Y$ where Y is any paracompact space. Recall from §1.2 that every metric space is paracompact. Therefore, the class of spaces Y now being considered is much larger than that of the last section. It is then natural to expect that the results will be more difficult to come by and this is, in fact, the case. Our main theorem consists of three sufficient conditions

on a space X which guarantee that the product space $X\times Y$ will be paracompact for every paracompact space Y. Furthermore, these three conditions are essentially independent of each other.

The question of paracompactness and normality in the product has been simplified greatly by a result obtained by H. Tamano. He was able to prove that paracompactness and normality were equivalent in the product space $X\times Y$ where Y is any paracompact space. Before stating this result, recall that a COMPACTIFICATION of a space X is an ordered pair (K,h) where K is a compact Hausdorff space and H is an embedding of X as a dense subset of K. The STONE-CECH COMPACTIFICATION, βX of X, is characterized among all other compactifications by the fact that every bounded, continuous, real-valued function on X has a continuous extension over βX.

Theorem 3.1. The following are equivalent for a space X:

(a) $X\times Y$ is normal for every paracompact space Y;

(b) $X\times Y$ is paracompact for every paracompact space Y.

Proof: (b) $\rightarrow$ (a). Every paracompact space is normal.

(a) $\rightarrow$ (b). Let $\beta(X\times Y)$ denote the Stone-Cech compactification of $X\times Y$. Since Y is paracompact, we have that $Y\times\beta(X\times Y)$ is paracompact. But $X\times Y$ is normal for every paracompact space Y. So $X\times[Y\times\beta(X\times Y)]$ is normal, which implies that $(X\times Y)\times\beta(X\times Y)$ is normal. Hence $X\times Y$ is paracompact by Tamano's theorem (§5.2).

This important result enables us to deal with the question of paracompactness and normality in the product simultaneously. However, note that the case being considered is fairly restricted. Essentially, the theorem says that if the product space $X \times Y$ is paracompact for <u>every</u> paracompact space Y, then these products are also normal, and vice-versa. We pose the following question: if X is a topological space, and Y is a particular paracompact space such that the product space $X \times Y$ is normal, then does it necessarily follow that $X \times Y$ is paracompact? As far as we know, this question is still open.

As stated previously, the main result of this section consists of three sufficient conditions on a space X which make the product space $X \times Y$ paracompact for every paracompact space Y. While there are other results available, the three we have chosen are considered to be the best available, and each will be discussed separately in the following subsections.

3.2. <u>Morita's Result</u>. A space X is said to be LOCALLY COMPACT if each point $x \in X$ has a base of compact neighborhoods. Note that every compact Hausdorff space is locally compact and that the open continuous image of a locally compact space is locally compact ([W], §18).

Morita's result states that if X is a paracompact space which is a countable union of locally compact closed subsets, then $X \times Y$ is paracompact for every paracompact space Y. To prove this result, we will need the following lemma.

<u>Lemma 3.2</u>. The following are equivalent for a paracompact space X:

(a) X is a countable union of locally compact closed subsets.

(b) X is a union of a σ - locally finite system of compact subsets.

Proof: (a) $\to$ (b) . Let $X = \bigcup_{i=1}^{\infty} C_i$ where each C_i is closed and locally compact. Then for each i , C_i is closed and hence paracompact since X is paracompact. Now C_i is locally compact, and so each $x \in C_i$ has a compact neighborhood G_{i_x} in C_i . Then $\{G_{i_x} \mid x \in C_i\}$ is a cover of C_i . Let $H_{i_x} = \mathrm{Int}_{C_i} G_{i_x}$. Then $\{H_{i_x}\}$ is an open cover of C_i , and since C_i is paracompact, there exists a closed locally finite refinement, say $\{K_{i\alpha}\}$. Each $K_{i\alpha}$ is closed and $K_{i\alpha} \subset G_{i_x}$ for some x , where G_{i_x} is compact. Hence $K_{i\alpha}$ is compact. Now let $\mathcal{K}_i = \{K_{i\alpha}\}$. Then $\mathcal{K} = \bigcup_{i=1}^{\infty} \mathcal{K}_i$ is a σ - locally finite system of compact subsets.

(b) $\to$ (a) . Suppose $X = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ where $\mathcal{B}_i = \{B_{i\alpha}\}$ is a locally finite collection of compact (and hence closed) subsets of X . Now the union of a locally finite collection of closed sets is closed ([W], p. 145). Let $B_i = \bigcup_{\alpha} B_{i\alpha}$. Then for each i , B_i is closed and locally compact (since each point of B_i has a neighborhood contained in the union of finitely many compact sets). Hence $X = \bigcup_{i=1}^{\infty} B_i$ is a countable union of locally compact subsets.

Theorem 3.3. Let X be a paracompact space which is a countable union of locally compact closed subsets. Then the product space $X \times Y$ is paracompact for every paracompact space Y .

Proof: Let X be a paracompact space which is a countable union of locally compact closed subsets. Then, by Lemma 3.2, there exists a σ-locally finite closed covering $\{A_{i\alpha} \mid \alpha \in \Omega_i, i = 1,2,\cdots\}$ of X such that $\{A_{i\alpha} \mid \alpha \in \Omega_i\}$ is locally finite in X for each i, and each subset $A_{i\alpha}$ is compact. Since X is paracompact (and normal), there are open subsets $L_{i\alpha}$, $\alpha \in \Omega_i$, $i = 1,2,\cdots$, such that

(1) $A_{i\alpha} \subset L_{i\alpha}$,

(2) $\{L_{i\alpha} \mid \alpha \in \Omega_i\}$ is locally finite in X for each i .

Let $\mathcal{M}$ be any open covering of $X \times Y$, where Y is a paracompact space. Then for each point $y \in Y$ and for each $A_{i\alpha}$, we can find an open neighborhood $V(y)$ of y in Y and a finite system $\{U_j \mid j = 1,2,\cdots,s\}$ of open subsets of X such that

(3) $$U_j \times V(y) \subset M_\lambda \quad \text{for some} \quad M_\lambda \in \mathcal{M},\ j = 1,\cdots,s \quad ,$$

(4) $$A_{i\alpha} \subset \bigcup_{j=1}^{s} U_j \subset L_{i\alpha} \quad .$$

This follows easily from the compactness of $A_{i\alpha}$. From (4) and the normality of $A_{i\alpha}$, we can find closed subsets F_j, $j = 1,\cdots,s$ of X such that

(5) $$A_{i\alpha} = \bigcup_{j=1}^{s} F_j \quad ; \quad F_j \subset U_j \ , \ j = 1,\cdots,s \quad .$$

Since X is normal, there exists open F_σ subsets U'_j, $j = 1,\cdots,s$ such that $F_j \subset U'_j \subset U_j$.

Now the family $\{V(y) \mid y \in Y\}$ forms an open covering of Y. Since Y is paracompact and normal, this covering can be refined by a locally finite covering of Y which consists of open F_σ subsets.

Thus, for each $A_{i\alpha}$ we can find a locally finite covering $\mathcal{H}(i,\alpha) = \{H(\lambda;i,\alpha) \mid \lambda \in \Lambda(i,\alpha)\}$ of Y be open F_σ subsets, and a family of finite systems $\mathcal{G}(\lambda;i,\alpha)$, $\lambda \in \Lambda(i,\alpha)$ consisting of open F_σ subsets of X such that

(6) $$A_{i\alpha} \subset \{G \mid G \in \mathcal{G}(\lambda;i,\alpha)\} \subset L_{i\alpha} \quad \text{for} \quad \lambda \in \Lambda(i,\alpha) \quad ,$$

(7) $$G \times X(\lambda;i,\alpha) \subset M_\gamma \quad \text{for some} \quad M_\gamma \in \mathcal{M} \ , \ G \in \mathcal{G}(\lambda;i,\alpha) \quad .$$

Let us put

(8) $$\mathcal{R}(i,\alpha) = \{G \times H(\lambda;i,\alpha) \mid G \in \mathcal{R}(\lambda;i,\alpha) \ ; \ \lambda \in \Lambda(i,\alpha) \quad ,$$

(9) $$\mathcal{R}_i = \cup\{\mathcal{R}(i,\alpha) \mid \alpha \in \Omega_i\} \quad , \quad \text{and}$$

(10) $$\mathcal{R} = \cup\{\mathcal{R}_i \mid i = 1,2,\cdots\} \quad .$$

Then the union of all the sets in $\mathcal{R}(i,\alpha)$ contains $A_{i\alpha} \times Y$. Hence the family $\mathcal{R}$ is an open covering of $X \times Y$, and from the construction, it is evident that $\mathcal{R}$ is a refinement of $\mathcal{M}$.

We will prove that $\mathcal{R}_i$ is locally finite in $X \times Y$ for each i. Let (x_o, y_o) be any point of $X \times Y$. Then there exists an open neighborhood U_o of x_o in X, such that U_o intersects only finitely many elements of $\{L_{i\alpha} \mid \alpha \in \Omega_i\}$. Let $\Gamma_o = \{\alpha \in \Omega_i \mid L_{i\alpha} \cap U_o \neq \phi\}$. Then

Γ_o is a finite set and it follows that $(U_o \times Y) \cap K = \phi$ for $K \in \mathcal{R}(i,\alpha)$ with $\alpha \in \Gamma_o$. For each $\alpha \in \Gamma_o$ we can find an open neighborhood V_α of y_o such that V_α intersects only finitely many elements of $\mathcal{H}(i,\alpha)$. Let us put $V_o = \cap\{V_\alpha \mid \alpha \in \Gamma_o\}$. Since Γ_o is a finite set, V_o is an open neighborhood of y_o in Y. Furthermore, $U_o \times V_o$ intersects only finitely many elements of $\mathcal{R}_i$. Hence $\mathcal{R}_i$ is locally finite.

Now each set of $\mathcal{G}(\lambda;i,\alpha)$ is an open F_σ set in X and each set of $\mathcal{H}(i,\alpha)$ is an open F_σ set in Y. Hence for each set $G \times H(\lambda;i,\alpha)$ of $\mathcal{R}(i,\alpha)$, there exists a non-negative continuous function ψ over $X \times Y$ such that

$$G \times H(\lambda;i,\alpha) = \{(x,y) \mid \psi(x,y) > 0\}$$

(let $\psi(x,y) = f(x) \cdot g(y)$ for $x \in X$, $y \in Y$ where $f : X \to [0,1]$ and $g : Y \to [0,1]$ are continuous maps such that $G = \{x \mid f(x) > 0\}$, $H(\lambda;i,\alpha) = \{y \mid g(y) > 0\}$). Then by Theorem 2.10, it follows that $\mathcal{R}$ is a normal covering of $X \times Y$. But $\mathcal{R}$ is a refinement of $\mathcal{M}$, so $\mathcal{M}$ is a normal covering of $X \times Y$. Hence $X \times Y$ is paracompact.

3.3. Katuta's[12] Result. A collection $\{A_\lambda \mid \lambda \in \Lambda\}$ of subsets of a space is called ORDER LOCALLY FINITE if we can introduce a total order $<$ in the index set Λ such that for each $\lambda \in \Lambda$, $\{A_\mu \mid \mu < \lambda\}$ is locally

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12. Recently, Katuta (Katu_2) has been able to prove several necessary and sufficient conditions which guarantee a paracompact product, $X \times Y$ where Y is any paracompact space.

finite at each point of A_λ . This definition provides the basis for our discussion of Katuta's theorem which states that if a regular space X has two coverings $\{C_\lambda \mid \lambda \in \Lambda\}$ and $\{U_\lambda \mid \lambda \in \Lambda\}$ such that

(i) C_λ is compact, U_λ is open, and $C_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$, and

(ii) $\{U_\lambda \mid \lambda \in \Lambda\}$ is order locally finite,

then the product space $X \times Y$ is paracompact for any paracompact regular space Y .

When X and Y are regular spaces, Katuta's condition covers the condition expressed in Morita's result. For suppose X is a paracompact regular space which is a countable union of locally compact closed subsets. Then there exists a σ - locally finite covering $\{C_\lambda \mid \lambda \in \Lambda\}$ of X such that each C_λ is compact. Moreover, since X is paracompact, there exists a σ - locally finite open covering $\{U_\lambda \mid \lambda \in \Lambda\}$ of X such that U_λ contains C_λ for each $\lambda \in \Lambda$. But, by Lemma 3.4 which follows immediately, a σ - locally finite collection is order locally finite. Hence, Katuta's result covers Morita's when X and Y are regular spaces.

We will need the following two lemmas to prove Katuta's theorem.

Lemma 3.4. Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be an order locally finite collection of subsets of a space X , and let $\{B_\alpha \mid \alpha \in \Omega_\lambda\}$ be a collection of subsets of A_λ which is locally finite in X for each $\lambda \in \Lambda$. Then the collection $\{B_\alpha \mid \alpha \in \Omega\}$ is order locally finite, where Ω is the disjoint union of the Ω_λ's . (In particular, a σ - locally finite collection is order locally finite.)

Proof: By definition of order locally finite, Λ has a total order $<$ such that for each $\lambda \in \Lambda$, $\{A_\mu \mid \mu < \lambda\}$ is locally finite at each point of A_λ. Now for each $\lambda \in \Lambda$, let $<_\lambda$ be any total order in Ω_λ. We will now define a total order $<<$ in Ω as follows: Let $\alpha_1, \alpha_2 \in \Omega$. Then $\alpha_1 \in \Omega_{\lambda_1}$ and $\alpha_2 \in \Omega_{\lambda_2}$ for some $\lambda_1, \lambda_2 \in \Lambda$. Now define

$$\alpha_1 << \alpha_2 \text{ if } \lambda_1 \neq \lambda_2 \text{ and } \lambda_1 < \lambda_2 \text{, or } \lambda_1 = \lambda_2 \text{ and } \alpha_1 <_\lambda \alpha_2 .$$

This is a total order on Ω.

Now let $x \in B_\alpha$ where $\alpha \in \Omega_\lambda$. Then $x \in A_\lambda$. Hence there exists a neighborhood $U(x)$ of x which intersects only finitely many A_μ for $\mu < \lambda$; let these be $A_{\mu_1}, \cdots, A_{\mu_m}$. Since for each $i = 1, \cdots, m$ the collection $\{B_\alpha \mid \alpha \in \Omega_{\mu_i}\}$ is locally finite in X, there exists a neighborhood $V_i(x)$ of x which intersects at most finitely many B_α for $\alpha \in \Omega_{\mu_i}$. Then the neighborhood $U(x) \cap V_1(x) \cap \cdots \cap V_m(x)$ of x intersects only finitely many B_η for $\eta << \alpha$. Hence the collection $\{B_\alpha \mid \alpha \in \Omega\}$ is order locally finite.

Lemma 3.5. A regular space X is paracompact if and only if any open covering of X has an order locally finite open refinement.

Proof: $\longrightarrow$. This part is clear.

$\longleftarrow$. Let $\mathcal{G}$ be an arbitrary open covering of X, and let $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be an order locally finite open refinement of $\mathcal{G}$. It is sufficient ([W], p. 146) to show that $\mathcal{G}$ has a locally finite refine-

ment.

By the definition of order locally finite, Λ has a total order $<$ such that for each λ, $\{U_\mu \mid \mu < \lambda\}$ is locally finite at each point of U_λ. For each $\lambda \in \Lambda$, define $V_\lambda = U_\lambda - \cup\{U_\mu \mid \mu < \lambda\}$. We will show that the collection $\mathcal{B} = \{V_\lambda \mid \lambda \in \Lambda\}$ is a locally finite covering of X. For any $x \in X$, x is contained in some $U_{\lambda_o} \in \mathcal{U}$. Since $\mathcal{U}$ is order locally finite, x is contained in only finitely many U_μ for $\mu < \lambda_o$, say $U_{\mu_1}, \cdots, U_{\mu_m}$ where $\mu_1 < \cdots < \mu_m$. Then clearly $x \in V_{\mu_1}$ and so $\mathcal{B}$ is a covering of X. Also, by assumption, we have that x has a neighborhood $W(x)$ which intersects only finitely many U_μ for $\mu < \lambda_o$. Then the neighborhood $W(x) \cap U_{\lambda_o}$ of x intersects only finitely many V_γ for $\gamma \in \Lambda$. So $\mathcal{B}$ is locally finite in X, and it is clear that $\mathcal{B}$ refines $\mathcal{G}$. Hence X is paracompact.

We will now state and prove Katuta's theorem.

<u>Theorem 3.6</u>. If a regular space X has two coverings $\{C_\lambda \mid \lambda \in \Lambda\}$ and $\{U_\lambda \mid \lambda \in \Lambda\}$ such that

(i) C_λ is compact, U_λ is open and $C_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$, and

(ii) $\{U_\lambda \mid \lambda \in \Lambda\}$ is order locally finite,

then the product space $X \times Y$ is paracompact for every paracompact regular space Y.

<u>Proof</u>: By Lemma 3.5, it is sufficient to show that any open covering of $X \times Y$ has an order locally finite open refinement. For each $\lambda \in \Lambda$ and

for each point $y \in Y$, we can find a finite collection $\{H_1, \cdots, H_m\}$ of open subsets of X and an open neighborhood $V(y)$ of $y \in Y$ such that $H_i \times V(y) \subset G_\gamma$ for some element $G_\gamma \in \mathcal{G}$, $i = 1, \cdots, m$, and $C_\lambda \subset \bigcup_{i=1}^{m} H_i \subset U_\lambda$. This follows easily since C_λ is compact. Then the collection $\{V(y) \mid y \in Y\}$ forms an open covering of Y. Since Y is paracompact, this covering has a locally finite open refinement.

Now for each element $\lambda \in \Lambda$, we can find a locally finite open covering $\mathcal{B}_\lambda = \{V_\alpha \mid \alpha \in \Omega_\lambda\}$ of Y and collections $\mathcal{H}_\alpha$, $\alpha \in \Omega_\lambda$, each of which consists of finitely many open subsets of X such that $C_\lambda \subset \cup\{H \mid H \in \mathcal{H}_\alpha\} \subset U_\lambda$ for $\alpha \in \Omega_\lambda$, and $H \times V_\alpha \subset G_\gamma$ for some $G_\gamma \in \mathcal{G}$, $H \in \mathcal{H}_\alpha$, $\alpha \in \Omega_\lambda$. By the construction, the collection $\{H \times V_\alpha \mid H \in \mathcal{H}_\alpha, \alpha \in \Omega_\lambda\}$ of subsets of $U_\lambda \times Y$ is locally finite in $X \times Y$.

However, the collection $\{U_\lambda \times Y \mid \lambda \in \Lambda\}$ is order locally finite since the collection $\{U_\lambda \mid \lambda \in \Lambda\}$ is order locally finite. Hence by Lemma 3.4, the collection $\{H \times V_\alpha \mid H \in \mathcal{H}_\alpha, \alpha \in \Omega_\lambda, \lambda \in \Lambda\}$ is order locally finite and covers $X \times Y$ since $\{C_\lambda \mid \lambda \in \Lambda\}$ is a covering of X. It also refines the cover $\mathcal{G}$. Hence $X \times Y$ is paracompact.

3.4. Ishii's Result. Ishii approaches the problem of a paracompact product from a different direction. Specifically, his result states that if X is the image under a closed continuous mapping f of a locally compact and paracompact Hausdorff space R, then the product space $X \times Y$ is paracompact for any paracompact space Y. This result is an immediate consequence of Lemmas 3.9 and 3.10. In order to establish Lemma 3.10, we

will need the following two lemmas which are due to K. Morita.

Lemma 3.7. Let $\mathcal{G} = \{G_\alpha \mid \alpha \in \Omega\}$ be a locally finite system of open sets in a normal space R, and $\mathcal{F} = \{F_\alpha \mid \alpha \in \Omega\}$ a system of closed sets of R such that $F_\alpha \subset G_\alpha$, $\alpha \in \Omega$. Then there exists a system of open sets U_α, $\alpha \in \Omega$, such that

(1) $\qquad F_\alpha \subset U_\alpha, \quad \overline{U}_\alpha \subset G_\alpha, \quad \alpha \in \Omega$, and

(2) $\qquad$ the system $\{\overline{U}_\alpha \mid \alpha \in \Omega\}$ is similar to the system $\mathcal{F}$.

Proof: Let us assume that the set of indices α consists of all (transfinite) ordinal numbers which are less than a fixed ordinal Ω_o. Denote by ϕ_1 the system of sets which can be expressed as finite intersections of sets from $\mathcal{F}$ and are disjoint from F_1. Then, since $\mathcal{F}$ is locally finite, ϕ_1 is locally finite. Let S_1 be the union of the sets from ϕ_1. Then S_1 is closed in R and $F_1 \cap S_1 = \phi$. Therefore, there is an open set U_1 such that $F_1 \subset U_1$, $\overline{U}_1 \subset G_1$, and $\overline{U}_1 \subset R - S_1$. If we construct a system $\mathcal{U}_1 = \{\overline{U}_1, F_2, \cdots\}$ by replacing F_1 in $\mathcal{F}$ by $\overline{U}_1$, it follows that the system $\mathcal{U}_1$ is similar to $\mathcal{F}$.

The proof can now be completed by transfinite induction. Suppose that for any β less than some fixed ordinal $\alpha < \Omega_o$, there exists an open set U_β such that $F_\beta \subset U_\beta$, $\overline{U}_\beta \subset G_\beta$ and the system $\mathcal{U}_\beta = \{\overline{U}_\gamma \mid \gamma \leq \beta, F_\gamma \mid \beta < \gamma < \Omega_o\}$ is similar to $\mathcal{F}$. Then the system $\{\overline{U}_\gamma \mid \gamma \leq \alpha, F_\gamma \mid \alpha \leq \gamma < \Omega_o\}$ is also locally finite and similar to $\mathcal{F}$, as is easily shown. Hence, by the above method, we can construct an open

set U_α such that $F_\alpha \subset U_\alpha$, $\overline{U}_\alpha \subset G_\alpha \alpha$ and the system $\mathcal{U}_\alpha = \{\overline{U}_\gamma \mid \gamma \leq \alpha , F_\gamma \mid \alpha < \gamma < \Omega_o\}$ is similar to $\mathcal{F}$. The system $\mathcal{U} = \{\overline{U}_\alpha \mid \alpha \in \Omega\}$ constructed in the above manner is similar to $\mathcal{F}$.

Lemma 3.8. Let $\mathcal{F} = \{F_\alpha \mid \alpha \in \Omega\}$ be a locally finite system of closed subsets in a paracompact space R . Then there exists a locally finite system of open sets G_α , $\alpha \in \Omega$, such that $F_\alpha \subset G_\alpha$.

Proof: For each point $p \in R$, there exists a neighborhood $U(p)$ such that $U(p)$ intersects only a finite number of sets of $\mathcal{F}$. Let us put $\mathcal{U} = \{U(p) \mid p \in R\}$. Since every open cover of a paracompact space has an open star-refinement ([W], p. 151), we can find an open star-refinement $\mathcal{B}$ of some open star-refinement $\mathcal{B}^*$ of $\mathcal{U}$. Let $G_\alpha = St.(F_\alpha,\mathcal{U})$. Then $\{G_\alpha\}$ satisfies the condition of the lemma. For if $St.(x,\mathcal{B}) \cap G_\alpha \neq \phi$, then we have $St.(x,\mathcal{B}^*) \cap F_\alpha \neq \phi$. Since $St.(x,\mathcal{B}^*)$ is contained in some $U(p)$, $St.(x,\mathcal{B})$ intersects only a finite number of sets G_α .

The following lemma is due to Ishii.

Lemma 3.9. Let $f : R \to X$ and $g : S \to Y$ be closed continuous maps of paracompact Hausdorff spaces R and S onto spaces X and Y , and let K be the set of irregular points[13] of $X \times Y$ with respect to the product mapping $h = f \times g$. If the projection $X' = \{X_\lambda \mid \lambda \in \Lambda\}$ of K to X is closed and discrete in X and if $R \times S$ is paracompact, then $X \times Y$ is para-

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13. (x,y) is said to be an IRREGULAR POINT if it does not satisfy the following: For any open subset U of $R \times S$ such that $f^{-1}(x)\ g^{-1}(y) \subset U$, there exists open subsets $G \subset R$ and $H \subset S$ such that $f^{-1}(x) \times g^{-1}(y) \subset G \times H \subset U$.

compact.

Proof: Since X and Y are the images under closed continuous mappings of paracompact Hausdorff spaces R and S, they are also paracompact Hausdorff spaces. Then, from Lemmas 3.7 and 3.8, it follows that for a closed discrete subset X' of X, there is a locally finite collection $\{N_\lambda \mid \lambda \in \Lambda\}$ of open subsets of X such that $x_\lambda \in N_\lambda$ and $\overline{N}_\lambda \cap \overline{N}_\mu = \phi$ for $\lambda \neq \mu$. Let $\mathcal{M} = \{M_\alpha \mid \alpha \in \Omega\}$ be any open covering of $X \times Y$. For each point (x_λ, y) there exists an open neighborhood $U_y(x_\lambda) \times V(y)$ of (x_λ, y) such that $U_y(x_\lambda) \times V(y)$ is contained in some M_α of $\mathcal{M}$ and $U_y(x_\lambda) \subset N_\lambda$. Since Y is a paracompact Hausdorff space, an open covering $\{V(y) \mid y \in Y\}$ of Y has a locally finite partition of unity subordinated to it; that is, there exists a family $\{g_\sigma^\lambda \mid \sigma \in \Gamma\}$ of real valued continuous functions on Y such that $0 \leq g_\sigma^\lambda(y) \leq 1$, $\sum_\sigma g_\sigma^\lambda(y) = 1$, $G_\sigma^\lambda = \{y \mid g_\sigma^\lambda(y) > 0\} \subset$ some $V(y)$, and $\{G_\sigma^\lambda\}$ is locally finite in Y. Hence for some fixed λ, we can find a family $\{U_\sigma(x_\lambda) \times G_\sigma^\lambda \mid \sigma \in \Gamma\}$ of open subsets of $X \times Y$ such that $U_\sigma(x_\lambda) \times G_\sigma^\lambda \subset$ some $U_y(x_\lambda)\ V(y)$ and $U_\sigma(x_\lambda) = \{x \mid f_\sigma^\lambda(x) > 0\}$ where $f_\sigma^\lambda : X \to [0,1]$ are continuous functions such that $f_\sigma^\lambda(x_\lambda) = 1$.

Now let $F_\lambda(x,y) = \sum_\sigma f_\sigma^\lambda(x)\ g_\sigma^\lambda(y)$, and $F(x,y) = \sum_\lambda F_\lambda(x,y)$.

Then $F_\lambda : X \times Y \to [0,1]$ and $F : X \times Y \to [0,1]$ are continuous functions and furthermore $\bigcup_\sigma [U_\sigma(x_\lambda) \times G_\sigma^\lambda] = \{(x,y) \mid F_\lambda(x,y) > 0\}$, and $F_\lambda(x_\lambda, y) = 1$ for every $y \in Y$. Let $P_\lambda = \{(x,y) \mid F_\lambda(x,y) > 0\}$, $Q_\lambda^1 = \{(x,y) \mid F_\lambda(x,y) \geq \frac{1}{2}\}$, $Q_\lambda^2 = \{(x,y) \mid F_\lambda(x,y) \geq \frac{1}{3}\}$ and $A = X \times Y - \bigcup_\lambda Q_\lambda^1$. Then we have

$Q^1_\lambda \subset Q^2_\lambda \subset P_\lambda = \underset{\sigma}{\cup}[U_\sigma(x_\lambda) \times G^\lambda_\sigma]$, $\underset{\lambda}{\cup} Q^1_\lambda = \{(x,y) \mid F(x,y) \leq \frac{1}{2}\}$, $\underset{\lambda}{\cup} Q^2_\lambda = \{(x,y) \mid F(x,y) \geq \frac{1}{3}\}$, and $A = \{(x,y) \mid F(x,y) < \frac{1}{2}\}$. Now from $h^{-1}(a) = \{(r,s) \mid F(h(r,s)) < \frac{1}{2}\}$, it follows that $h^{-1}(A)$ is an open F_σ subset of $R \times S$. Hence $h^{-1}(A)$ is paracompact as a subspace of $R \times S$. Also, it is easily shown that the mapping $h^* = h \mid h^{-1}(A)$ is closed. In fact, let F be a closed subset of $R \times S$. Then $\overline{h(F)} - h(F)$ contains only irregular points of $X \times Y$ with respect to h ; that is, $\overline{h(F)} - f(F) \subset K$. Since $K \cap A = \phi$, we have $h(h^{-1}(A) \cap F) = A \cap h(F) = A \cap \overline{h(F)}$, which shows that $A \cap h(F)$ is closed in A . Therefore, A is paracompact as a subspace of $X \times Y$ since it is the image of a paracompact space $h^{-1}(A)$ under the closed continuous mapping $h^* : h^{-1}(A) \to A$. Hence an open covering $\mathcal{U} = \{A \cap M_\alpha \mid \alpha \in \Omega\}$ of A admits a locally finite open refinement $\{G_\alpha \mid \alpha \in \Omega\}$ in A . For every $\alpha \in \Omega$, let us put $L_\alpha = G_\alpha - \underset{\lambda}{\cup} Q^2_\lambda$. Then $\{L_\alpha \mid \alpha \in \Omega\}$ is a family of open subsets of $X \times Y$ which is locally finite in $X \times Y$. If we let $\mathcal{N} = \{L_\alpha , U_\sigma(x_\lambda) \times G^\lambda_\sigma \mid \alpha \in \Omega , \sigma \in \Gamma , \lambda \in \Lambda\}$, then $\mathcal{N}$ is a locally finite open refinement of $\mathcal{M}$. Hence $X \times Y$ is paracompact.

The final lemma needed for the proof of Ishii's theorem is due to Morita.

<u>Lemma 3.10</u>. Let f be a closed continuous mapping of a locally compact and paracompact Hausdorff space R onto another space X . If we denote by X' the set of all points $x \in X$ such that $f^{-1}(x)$ is not compact, then X' is a closed discrete subset of X .

Proof: It is sufficient to prove that $\{f^{-1}(x) \mid x \in X'\}$ is a discrete collection of closed sets in R. Since X is locally compact, we can do this by showing that any compact set C intersects only a finite number of sets $f^{-1}(x)$ for $x \in X'$. Suppose that there exist a countably infinite number of points y_i, $i = 1,2,\cdots$ of R such that $y_i \in C \cap f^{-1}(x_i)$, $x_i \in X'$, $i = 1,2,\cdots$; $y_i \neq y_j$ for $i \neq j$.

Since C is compact, there exists a limit point y_o of the set $\{y_i \mid i = 1,2,\cdots\}$. We may assume that $f(y_o) \neq x_i$, $i = 1,2,\cdots$; if $f(y_o) = x_i$ for some i we have only to replace $\{y_j\}$ by $\{y_j \mid j \neq i\}$. Putting $x_o = f(y_o)$, we have

(1) $$x_o \in X' \; ; \; x_o \neq x_i \quad \text{for} \quad i = 1,2,\cdots .$$

To prove (1), suppose that $x_o \in X - X'$. Then $f^{-1}(x_o)$ is compact. Since R is locally compact, there exists an open set L such that $\overline{L}$ is compact and $f^{-1}(x_o) \subset L$. If we put $M = X - f(R-L)$, then M is an open set in X and $y_o \in f^{-1}(x_o) \subset f^{-1}(M) \subset L$. The point y_o is a limit point of $\{y_i\}$ and hence $y_i \in f^{-1}(M)$ for some i. Therefore for such i we have $f^{-1}(x_i) \subset f^{-1}(M) \subset L$. Thus $f^{-1}(x_i)$ must be compact, but this contradicts the assumption that $x_i \in X'$. Hence (1) follows.

By the assumption of the theorem, R is paracompact and locally compact, and hence there exists a locally finite open covering $\{G_\alpha \mid \alpha \in \Omega\}$ of R such that $\overline{G}_\alpha$ is compact for every α. If we put

(2) $$\Gamma = \{\alpha \mid G_\alpha \cap f^{-1}(x_o) \neq \phi\} ,$$

then Γ is an infinite set since $f^{-1}(x_o)$ is not compact. Let

(3) $$G = \cup\{G_\alpha \mid \alpha \in \Gamma\} \ , \quad V_o = X - f(R-G) \ .$$

Then V_o is open and $f^{-1}(x_o) \subset f^{-1}(V_o) \subset G$.

The set of all points y_i which belong to $f^{-1}(V_o)$ consists of an infinite number of points since R is a T_1 - space. We will denote these points by y_{k_i} , $i = 1,2,\cdots$. Then y_o is clearly a limit point of the set $\{y_{k_i}\}$. Therefore, if we put $D = \{x_{k_i} \mid i = 1,2,\cdots\}$ we have

(4) $$x_o \in \overline{D} - D \ .$$

Now we have $y_{k_i} \in f^{-1}(V_o)$ and hence $x_{k_i} \in V_o$. Thus

(5) $$f^{-1}(x_{k_i}) \subset f^{-1}(V_o) \subset G \ , \quad i = 1,2,\cdots \ .$$

In view of (3) and (5) we can find points y'_{k_i} of R and elements α_i of Γ such that

(6) $$y'_{k_1} \in f^{-1}(x_{k_1}) \cap G_{\alpha_1} \quad \text{and} \quad y'_{k_i} \in f^{-1}(x_{k_i}) \cap \left(R - \bigcup_{j=1}^{i-1} G_{\alpha_j}\right) \cap G_{\alpha_i} \ ,$$

$$i = 2,3,\cdots \ .$$

Indeed, since $f^{-1}(x_{k_i})$ is not compact, we have $f^{-1}(x_{k_i}) \cap \left(R - \bigcup_{j=1}^{i-1} G_{\alpha_j}\right) \neq \phi$ for any finite number of sets $G_{\alpha_1},\cdots,G_{\alpha_{j-1}}$, and hence these y'_{k_i} , α_i can be found by induction.

Since $y'_{k_i} \in G_{\alpha_i}$, $\alpha_i \neq \alpha_j$ for $i \neq j$ and $\{G_\alpha \mid \alpha \in \Gamma\}$ is locally finite, the set $\{y'_{k_i} \mid i = 1,2,\cdots\}$ is a closed subset of R . Therefore $D = \{x_{k_i}\}$ is closed in X since f is a closed map. However (4) shows that D is not closed in X . Hence the assertion is proved by contradiction.

The following theorem gives Ishii's result for paracompact products.

Theorem 3.11. Let X be the image under a closed continuous mapping f of a locally compact and paracompact Hausdorff space R and let Y be a paracompact space. Then the product $X \times Y$ is paracompact.

Proof: The product space $R \times Y$ of a locally compact and paracompact Hausdorff space R with a paracompact space Y is paracompact. Therefore, Theorem 3.11 follows immediately from Lemma 3.9 and 3.10.

Remark: Essentially, the three conditions discussed in the last three sections are independent of each other. As stated in §3.3, Katuta was able to show that his result covers Morita's when X and Y are regular spaces. He was also able to show that his result is not covered by either of the other two. Ishii was able to show that his condition is not covered by Morita's. Finally, J. Suzuki showed that Katuta's result does not cover Ishii's.

3.5. Other Results. Among the other available sufficient conditions are the following, which will be stated without proof.

Theorem 3.12. The product space $X \times Y$ is paracompact for every paracompact space Y in each of the following cases:

(a) X is compact (J. Dieudonné, cited in [Mo_5]).

(b) X is σ - compact and regular [Mi_1].

(c) X is paracompact and locally compact (K. Morita, cited in [Mo_5]).

(d) X is the closed continuous image of a paracompact and perfectly normal Hausdorff space [Ts].

As stated previously, there are no necessary and sufficient conditions.

§4. Compact Metric Spaces.

4.1. Introduction. In §2 we dealt with the problem of a paracompact or normal product $X \times Y$ where Y was any metric space. We would now like to answer the question: what spaces X have a normal or paracompact product with every compact metric space Y ? The class of spaces now being considered is much smaller than that of §2; hence the conditions provided in §2 are all sufficient in the present case. However, because we have placed a further restriction on the spaces Y , it is reasonable to expect that better results will be available, and this, in fact, is the case.

4.2. Main Theorems. The two theorems which we will present as the main results of this section are both well known. The first theorem provides an answer for paracompact products while the second deals with normal products.

Theorem 4.1. Let Y be a compact metric space. Then the product $X \times Y$ is paracompact if and only if X is paracompact.

Proof: $\longrightarrow$. X is an F_σ subset of $X \times Y$ and any F_σ subset of paracompact space is again paracompact.

$\longleftarrow$. The product of a paracompact space with a compact space is always paracompact.

The second theorem is due to C.H. Dowker. The proof will not be given here since it is well known ([W], p. 158).

Theorem 4.2. Let Y be a compact metric space. Then the product $X \times Y$ is normal if and only if X is countably paracompact and normal.

4.3. Dowker's Conjecture. In view of the above theorem, we know that $X \times I$ where $I = [0,1]$ is normal if and only if X is countably paracompact and normal (= binormal). Dowker wondered, in the paper in which he proved Theorem 4.2 [Do], whether every normal space was binormal; the assertion that this was so came to be known as Dowker's Conjecture. A counterexample to Dowker's Conjecture (that is, a normal space which is not binormal) is called a Dowker space. It is only recently (1971) that M.E. Rudin has been able to show the existence of a Dowker space using only set theoretic axioms through the axiom of choice [R] . Hence, Dowker's Conjecture is false.

§5. Compact Spaces.

5.1. Introduction. Let Y be a compact space. In this section we are interested in answering the question: what spaces X have a paracompact or normal product with every compact space Y ? One result which is immediate and obvious is that the product space $X \times Y$ is paracompact for every compact space Y if and only if X is paracompact. We will show in our main result that these conditions are equivalent to $X \times Y$ having a normal product for every compact space Y .

5.2. Tamano's Theorem. In order to prove the main theorem, we will have to make reference to Tamano's famous theorem which gives an important if not conclusive characterization of paracompactness. Recall that βX denotes the Stone-Cech compactification of a space X (§3.1). The theorem states that for a Tychonoff space, X is paracompact if and only if $X \times \beta X$ is normal. We will state this theorem in its original form without proof (see [W], p. 154 for proof).

Theorem 5.1. The following are equivalent for a Tychonoff space X :

(a) $X \times \beta X$ is normal.

(b) for each compact $F \subset \beta X - X$, there is a locally finite open cover $\{U_\lambda \mid \lambda \in \Lambda\}$ of X such that $(Cl_{\beta X} U_\lambda) \cap F = \phi$ for each $\lambda \in \Lambda$.

(c) X is paracompact.

5.3. Main Theorem. We can now state and prove our main result in this section. The proof of this theorem is simplified greatly through the use of Tamano's theorem.

Theorem 5.2. The following are equivalent for a compact space Y .

(a) $X \times Y$ is paracompact for every compact space Y .

(b) X is paracompact.

(c) $X \times Y$ is normal for every compact space Y .

Proof: (a) $\rightarrow$ (c) . Every paracompact space is normal.

(c) $\rightarrow$ (b) . Since $X \times Y$ is normal, it follows that X must be normal, and hence Tychonoff (since all spaces are assumed to be Hausdorff). Therefore we may apply Tamano's theorem. By assumption, $X \times Y$ is normal for every compact space Y . In particular, $X \times \beta X$ is normal and so X is paracompact by Theorem 5.1.

(b) $\rightarrow$ (a) . The product of a paracompact space and a compact space is always paracompact.

CHAPTER II

Paracompactness and Normality in Large Products

§6. Introduction. Chapter I dealt with the question of paracompactness and normality in the product of two topological spaces. In this chapter, we will be concerned with providing conditions under which larger finite and countable products will be paracompact or normal. Obviously since the product of even two paracompact (normal) spaces need not be paracompact (normal), larger products of paracompact (normal) spaces will often fail to be paracompact (normal).

In comparison with the amount of research that has been done on the product of two spaces, the amount of research on large paracompact or normal products is small. In §7, we will consider three theorems which provide sufficient conditions that guarantee that a countable product of paracompact spaces will again be paracompact, and hence normal. There is only one theorem available for normal products, and this appears in §8. Finally, in §9, we will devote considerable space to discussing a paper by E. Michael in which he constructs a number of counterexamples to refute some plausible conjectures. These examples demonstrate effectively the unpredictability of higher powers of a space X. For example, he provides an example of a space Y such that Y^n is paracompact for all integers n, but $Y^{\aleph_0}$ is not normal.

§7. Paracompact Products. The first result we will prove is due to A. Okuyama; the proof we present is K. Morita's. This theorem provides a

sufficient condition which guarantees that a countable product of topological spaces will be paracompact.

Theorem 7.1. If $X_1, X_2, \cdots$ are topological spaces, and if for all integers n, $\prod_{i=1}^{n} X_i$ is a paracompact space in which every open subset is an F_σ (= perfectly normal), then $\prod_{i=1}^{\infty} X_i$ is a paracompact space.

Proof: Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Omega\}$ be an open covering of $\prod_{i=1}^{\infty} X_i$. Without loss of generality, we may assume that for each $U_\alpha \in \mathcal{U}$,
$U_\alpha = U_\beta^1 \times \cdots \times U_\gamma^i \times \cdots U_\delta^{n_\alpha} \times \prod_{i>n_\alpha} X_i$ for some n_α, where U_γ^i is an open subset of X_i for each i $(1 \leq i \leq n_\alpha)$. Let
$\mathcal{U}_k = \{U_\alpha \mid \alpha \in \Omega , n_\alpha = k\}$, $\mathcal{U}_k' = \{U_\alpha' \mid U_\alpha \in \mathcal{U}_k\}$ where
$U_\alpha' = \{(x_1,\cdots,x_k) \mid (x_1,\cdots,x_k,\cdots) \in U_\alpha \in \mathcal{U}_k\}$, and $Y_k = \cup\{U_\alpha' \mid U_\alpha' \in \mathcal{U}_k'\}$.
Then Y_k is open in $X_1 \times \cdots \times X_k$. Since $X_1 \times \cdots \times X_k$ is perfectly normal and paracompact, there exists a σ - locally finite collection
$\mathcal{B}_k' = \{V_\lambda' \mid \lambda \in \Lambda_k\}$ of open subsets of $X_1 \times \cdots \times X_k$ whose union covers Y_k and which is a refinement of $\mathcal{U}_k \cap Y_k$. If we let
$V_\lambda = \{(x_1,\cdots,x_k,\cdots) \mid (x_1,\cdots,x_k) \in V_k'\}$ for each $\lambda \in \Lambda_k$, then it is readily seen that $\{V_\lambda \mid \lambda \in \Lambda_k , k = 1,2,\cdots\}$ is a σ - locally finite open covering of $\prod_{i=1}^{\infty} X_i$ which is a refinement of $\mathcal{U}$. Hence $\prod_{i=1}^{\infty} X_i$ is paracompact.

The next two theorems in this section are due to Z. Frolik and, like Theorem 7.1, they express a condition which is sufficient to make a countable product of topological spaces paracompact. First, recall from

§2.2 that a completely regular space X is said to TOPOLOGICALLY COMPLETE IN THE SENSE OF CECH if X is a G_δ subset of its Stone-Cech compactification βX. It is worth noting that if X is a complete space and X is dense in Y, then X is a G_δ subset of Y. Frolik also gives the following equivalent condition for complete. A completely regular space X is COMPLETE if and only if there exists a sequence $\{\mathcal{U}_n\}$ of open coverings of X satisfying the following condition: if $\mathcal{U}$ is a family of subsets of X having the finite intersection property and if $\mathcal{U} \cap \mathcal{U}_n \neq \phi$ for every integer n, then the intersection of the closures of sets from $\mathcal{U}$ is non-empty. Every sequence $\{\mathcal{U}_n\}$ of open coverings satisfying this condition is called complete.

We will also need the following definition. A mapping f from a space X to a space Y is said to be a PERFECT MAPPING if f is continuous, closed, and $f^{-1}(y)$ is compact in X for every $y \in Y$. It is easily verified that the product of perfect mappings is again a perfect map.

The following lemma due to Frolik will be needed in the proof of his theorems.

<u>Lemma 7.2</u>. Let X be a completely regular space. Then X is paracompact and complete if and only if there exists a perfect mapping of X onto a complete metric space.

<u>Proof</u>: $\longrightarrow$. X is complete, so there exists a complete sequence $\{\mathcal{U}_n\}$ of open coverings of X. Since X is paracompact, every $\mathcal{U}_n$ is an open normal covering. Therefore, we can construct a normal sequence $\{\mathcal{B}_n\}$ of

open coverings such that every $\mathcal{B}_n$ refines $\mathcal{U}_n$. It is clear that the sequence $\{\mathcal{B}_n\}$ is complete.

Now there exists a pseudometric ψ on X such that

(a) for every integer n , there exists an $\epsilon > 0$ such that the coverings consisting of all open spheres of radius ϵ refines $\mathcal{U}_n$, and

(b) for every $\epsilon > 0$, there exists an integer n such that $\mathcal{U}_n$ refines the covering consisting of all open ψ - spheres of radius ϵ .

Finally, there exists a mapping f of X onto a metric space (Y,ρ) such that $\psi(x,y) = \rho(f(x),f(y))$ for all x and y in X . Clearly f is a continuous mapping. To prove that f is a closed mapping, it is sufficient to show that whenever y is an accumulation point of $f[M]$, then there exists an accumulation point x of M such that $x \in f^{-1}(y)$. Let $\mathcal{M}$ be a maximal family of subsets of X having the finite intersection property and containing all sets of the form $M \cap f^{-1}(U)$ where U runs over all neighborhoods of y . According to (a) above, we know $\mathcal{M} \cap \mathcal{B}_n \neq \phi$ for all integers n . Thus $\bigcap_{M \in \mathcal{M}} \overline{M} \neq \phi$ and evidently is contained in $f^{-1}(y)$. To complete the proof that f is a perfect mapping, we must show that $f^{-1}(y)$ is compact for any $y \in Y$. To prove this, it is sufficient to note that for every maximal family of subsets of X having the finite intersection property and containing the set $f^{-1}(y)$, we have by (a) that $\mathcal{M} \cap \mathcal{B}_n \neq \phi$ for every integer n .

Therefore we have that f is a perfect mapping of X onto Y , and it is easy to see that (Y,ρ) is a compact metric space.

$\Longleftarrow$. Now suppose that there exists a perfect mapping f of X onto a complete metric space. To show that X is paracompact, let $\{U_\alpha \mid \alpha \in \Omega\}$ be any open covering of X . Consider any point $x \in X$. Then $x \in U_\alpha$ for some $\alpha \in \Omega$. Since f is a perfect mapping, there exists an open neighborhood V_x of $f(x)$ in Y such that $x \in f^{-1}(V_x) \subset U_\alpha$. Now consider $\{V_x \mid x \in X\}$. This is an open covering, and since X is a metric space (and hence paracompact), there exists an open locally finite refinement, say $\{K_\gamma \mid \gamma \in \Gamma\}$. We claim that $\{f^{-1}(K_\alpha) \mid \gamma \in \Gamma\}$ is an open locally finite refinement of $\{U_\alpha \mid \alpha \in \Omega\}$. Firstly, each $f^{-1}(K_\gamma)$ is open since f is a continuous mapping, so $\{f^{-1}(K_\gamma) \mid \gamma \in \Gamma\}$ is an open cover of X . Next, this cover is locally finite. For suppose not. Then there exists an $x \in X$ such that x is contained in infinitely many $f^{-1}(K_\gamma)$'s . But then $f(x)$ is contained in infinitely many K_γ 's, which is a contradiction since $\{K_\gamma \mid \gamma \in \Gamma\}$ is locally finite. Finally $\{f^{-1}(K_\gamma) \mid \gamma \in \Gamma\}$ is a refinement of $\{U_\alpha \mid \alpha \in \Omega\}$ since for each $\gamma \in \Gamma$, $K_\gamma \subset V_p$ for some $p \in Y$. But then $f^{-1}(K_\gamma) \subset f^{-1}(V_p) \subset U_\alpha$ for some $\alpha \in \Omega$. Hence we have that $\{f^{-1}(K_\gamma) \mid \gamma \in \Gamma\}$ is an open locally finite refinement of $\{U_\alpha \mid \alpha \in \Omega\}$, so X is paracompact.

We must now show that X is complete; that is we must show that X is a G_δ subset of βX . We have that Y is complete and that f is a perfect map from X onto Y . We can extend f to a continuous map F from βX to βY ([W], Theorem 19.5, p. 137). Now we claim that $F(\beta X-X) \subset \beta Y-Y$. Suppose to the contrary that there exists a point $p \in \beta X - X$ such that $F(p) = y \in Y$. Since f is a perfect map we know

that $H = f^{-1}(y)$ is a compact subset of X. Now let V be a neighborhood of H in βX whose closure does not meet p. Such a set V exists because of regularity, and we may assume that $V \cap X$ is saturated (that is, a union of inverse images of points in Y under f). Then $f(V \cap X)$ is a neighborhood of $y \in Y$ and so $F(V \cap X) = Y \cap W$ where W is a neighborhood of y in βY. But F is continuous at p, so there exists a neighborhood S of p in βX such that $F(S) \subset W$. However, S necessarily contains points of $X-V$, and $f = F \mid X$ takes such points outside W. Therefore, we have a contradiction and so $F(\beta X - X) \subset \beta Y - Y$.

Now Y is a G_δ subset of βY. Therefore there exists open subsets G_i of βY such that $Y = \bigcap_{i=1}^{\infty} G_i$. Since F is continuous, it follows that $F^{-1}(G_i)$ is open in βX for every i, and furthermore, $X = \bigcap_{i=1}^{\infty} F^{-1}(G_i)$. For suppose that there exists an $x \in \beta X - X$ such that $x \in \bigcap_{i=1}^{\infty} F^{-1}(G_i)$. Then we know that $x \in F^{-1}(G_i)$ for every i and so $F(x) \in G_i$ for every i. Then $F(x) \in \bigcap_{i=1}^{\infty} G_i = Y$, which is a contradiction since $F(\beta X - X) \subset \beta Y - Y$. Hence X is a G_δ subset of βX and it follows that X is complete.

We can now prove the two product theorems due to Frolik.

<u>Theorem 7.3</u>. Let X_i be a paracompact and complete space for every i. Then $\prod_{i=1}^{\infty} X_i$ is paracompact and complete.

<u>Proof</u>: X_i is paracompact and complete for every integer i. Therefore,

by Lemma 7.2, there exists a perfect map f_i from X_i onto some complete metric space Y_i. Consider $\prod_{i=1}^{\infty} X_i$ and $f(\{x_i\}) = (\{f_i(x_i)\})$. Then f is a perfect mapping from $\prod_{i=1}^{\infty} X_i$ onto $\prod_{i=1}^{\infty} Y_i$, and $\prod_{i=1}^{\infty} Y_i$ is a complete metric space ([W], p. 178 and p. 180). Hence, by Lemma 7.2, $\prod_{i=1}^{\infty} X_i$ is paracompact and complete.

Theorem 7.4. Let X_i be a metric space or a complete paracompact space for every integer i. Then $\prod_{i=1}^{\infty} X_i$ is paracompact.

Proof: We know that the countable product of metric spaces is again a metric space, and that the countable product of paracompact and complete spaces is again paracompact and complete. Then what we have is the product of a metric space with a paracompact and complete space. But a paracompact and complete space is a P - space (§2.2) and we know that the product of a paracompact P - space with a metric space is again paracompact (Theorem 2.19). Hence $\prod_{i=1}^{\infty} X_i$ is paracompact.

§8. Normal Products. Obviously, each of the products in Theorems 7.3 and 7.4 is normal. The result for normal products which we present in this section is due to Katetov. Recall that a space is said to be PERFECTLY NORMAL if and only if it is normal and every open subset is an F_σ, or equivalently, if and only if for every pair of disjoint closed subsets G, H of X, there exists a continuous function $f : X \to [0,1]$ such that $G = f^{-1}(0)$ and $H = f^{-1}(1)$.

<u>Theorem 8.1</u>. Let X_i be a topological space for every i. If for all integers i, $\prod_{i=1}^{n} X_i$ is a normal space in which every open subset is an F_σ, then $\prod_{i=1}^{\infty} X_i$ is normal and every open subset is an F_σ.

<u>Proof</u>: Let $A \subset \prod_{i=1}^{\infty} X_i$ be closed, and let π_n denote the projection of $\prod_{i=1}^{\infty} X_i$ onto $\prod_{i=1}^{n} X_i$. There exists a continuous function $g_n(y)$ on $\prod_{i=1}^{n} X_i$ such that $0 \leq g_n(y) \leq 1$ for all y and $g_n(y) = 0$ if and only if $y \in \overline{\pi_n(A)}$. For $x \in \prod_{i=1}^{\infty} X_i$, let us put $f_n(x) = g_n(\pi_n(x))$ and $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x)$. Then $f(x)$ is continuous on $\prod_{i=1}^{\infty} X_i$ and $0 \leq f(x) \leq 1$ for every $x \in \prod_{i=1}^{\infty} X_i$. Also $f(x) = 0$ for all $x \in A$, and for $x \in \prod_{i=1}^{\infty} X_i - A$ and some convenient m, we have $\pi_m(x) \notin \overline{\pi_m(A)}$ and so $f_m(x) > 0$ which implies that $f(x) > 0$. Therefore we have that $f(x) = 0$ if and only if $x \in A$, which completes the proof. (A well known theorem due to Urysohn states that a space X is perfectly normal if and only if there exists for every closed $A \subset X$ a continuous function f such that $f(x) = 0$ if and only if $x \in A$.)

§9. <u>Counterexamples</u>. In a remarkable paper entitled "Paracompactness and the Lindelöf Property in Finite and Countable Cartesian Products", Michael provides some excellent examples to demonstrate the unpredictability of higher powers of topological spaces. Examples 1 and 2 are done in detail while Example 3 and 4 are merely described. Anyone wishing to the detail of the latter should refer to the above paper.

Example 1. There exists a topological space Y such that Y^n is paracompact for every integer n, but Y^{χ_0} is not normal.

Y is the space obtained from the reals $\underset{\sim}{R}$ with its usual topology by making the subset P of irrationals discrete. Open subsets of Y are of the form $U \cup T$ where U is open in $\underset{\sim}{R}$ and $T \subset P$. This space is often referred to as the scattered line. The space Y has the following properties:

(1) Y is regular: This is clear since any closed set in Y must be of the form $Y - (U \cup T)$ where U is open in $\underset{\sim}{R}$ and $T \subset P$. Consider any point $y \notin Y - (U \cup T)$. Then two cases arise: (a) $y \in U$: in this case we can find an open interval (a,b) such that $y \in (a,b) \subset U$. Then the open set $(-\infty,a) \cup (b,\infty)$ is disjoint from (a,b) and contains $Y - (U \cup T)$. (b) $y \in T$: in this case, $\{y\}$ is an open set containing y and $(-\infty,y) \cup (y,\infty)$ is an open set disjoint from $\{y\}$ and containing $Y - (U \cup T)$. Hence Y is regular.

(2) $Y \times \underset{\sim}{P}$ is not normal: This also is clear. If we let Q denote the set of rational points in Y and U the set of irrational points in Y, then $A = Q \times \underset{\sim}{P}$ and $B = \{(x,x) \mid x \in U\}$ are two disjoint closed subsets of $Y \times \underset{\sim}{P}$ which cannot be separated by disjoint open sets. Suppose that V is a neighborhood of B in $Y \times \underset{\sim}{P}$. We will show that V contains an element of A. For each integer n, let $U_n = \{x \in U \mid (\{x\} \times S_{\frac{1}{n}}(x)) \subset V\}$, where $S_{\frac{1}{n}}(x)$ represents the sphere of radius $\frac{1}{n}$ with center x. Then U_n covers U for each n, and since U is not an F_σ in Y, there exists an integer k such that

$\overline{U}_k \cap Q \neq \phi$. Pick $y \in \overline{U}_k \cap Q$ and $p \in \underset{\sim}{P}$ such that $|y-p| < \frac{1}{2k}$. Then $(y,p) \in A$. Now we need only show that any rectangular neighborhood $R\times S$ of (y,p) intersects V . Pick $y' \in R \cap U_k$ so that $|y'-y| < \frac{1}{2k}$. Then $(y',p) \in R\times S$. Also, it follows that $|y'-p| \leq |y'-y| + |y-p| < \frac{1}{k}$. Hence $(y',p) \in V$ since $y' \in U_k$. Therefore, $Y\times\underset{\sim}{P}$ is not normal.

(3) The set of non-isolated[14] points of Y is Q and hence is countable.

We will now show that Y^n is paracompact for every integer n . This follows immediately from Theorems 9.2 and 9.3. However, before stating and proving these theorems, we must establish the following lemma. Recall that a continuous function r from a space X onto a subspace A of X is called a RETRACTION of X onto A if and only if $r|A$ is the identity map on A . Then A is called a RETRACT of X .

Lemma 9.1. Let X be a space, and let $B \subset X$ with $X - B$ countable. Then for every integer n , $X^n - B^n$ is the union of countably many subsets, each of which is a retract of X^n and homeomorphic to X^{n-1} .

Proof: For each integer $i \leq n$ and each element $a \in X-B$, let $Z_{i,a} = \{x \in X^n \mid x_i = a\}$. There are countably many such sets and it is easily verified that they have all the required properties.

Theorem 9.2. If X is a regular space with at most countably many non-isolated points, then X^n is paracompact for every integer n .

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14. A point x is said to be ISOLATED if and only if $\{x\}$ is open.

Proof: We will prove this theorem by induction. Let X^o be a one point set. Then it is clear that the assertion is true for $n = 0$. Now assume that X^{n-1} is paracompact and we will show that X^n is paracompact. Denote by B the set of isolated points of X. Let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of X^n. By Lemma 9.1, there are retracts Z_i of X^n which are homeomorphic to X^{n-1} such that $X^n - B^n = \bigcup_{i=1}^{\infty} Z_i$. Now each Z_i is paracompact by our inductive hypothesis. So for each i there exists a locally finite relatively open (with respect to Z_i) refinement $\{V_{\lambda,i} \mid \lambda \in \Lambda\}$ of $\{U_\lambda \cap Z_i \mid \lambda \in \Lambda\}$. Let $r_i : X^n \to Z_i$ be the canonical retraction for each integer i and let $W_{\lambda,i} = r_{\lambda,i}^{-1}(V_{\lambda,i}) \cap U_\lambda$, $\lambda \in \Lambda$, $i \in N$. Then $\mathcal{W} = \{W_{\lambda,i} \mid \lambda \in \Lambda, i \in N\}$ is a σ - locally finite collection of open subsets of X^n which covers $X^n - B^n$. But then $\mathcal{W} \cup \{\{x\} \mid x \in X^n - \cup\mathcal{W}\}$ is a σ - locally finite refinement of $\{U_\lambda \mid \lambda \quad \Lambda\}$. Since X^n is regular, it follows that X^n is paracompact ([Du], Theorem 2.3, p. 163).

Clearly the space Y described previously satisfies the conditions of Theorem 9.2. Therefore Y^n is paracompact for every integer n. We will now show that Y^{χ_o} is not normal.

Theorem 9.3. If X is a space for which X^{χ_o} is normal, then $X \times \underset{\sim}{P}$ is normal.

Proof: Since a countably compact and normal space is a P - space, it follows from Theorem 2.18 that the product of a countably compact and normal space with a metric space is normal. Therefore, if X is countably compact, then $X \times \underset{\sim}{P}$ is normal.

If X is not countably compact, then X has a closed subset homeomorphic to $\underset{\sim}{N}$. Hence $X^{\aleph_0}$ has a closed subset which is homeomorphic to $\underset{\sim}{N}^{\aleph_0}$, and hence to $\underset{\sim}{P}$. But $X^{\aleph_0}$ is homeomorphic to $X \times X^{\aleph_0}$, so $X^{\aleph_0}$ has a closed subset homeomorphic to $X \times \underset{\sim}{P}$. Hence, $X \times \underset{\sim}{P}$ is normal.

We have already shown that $Y \times \underset{\sim}{P}$ is not normal for the space Y constructed. Hence $Y^{\aleph_0}$ cannot be normal.

Example 2. (Assumes the continuum hypothesis[15].) There exists a regular space Y such that Y^n is Lindelöf[16] for every integer n, but $Y^{\aleph_0}$ is not normal.

Let X be the space described in Example 1; that is, X is the space obtained from the reals $\underset{\sim}{R}$ by making the subset P of irrationals discrete. We will show that there exists a subset Y of X such that Y^n is Lindelof for every integer n. We will need the following two lemmas.

Lemma 9.4. Let Y be a topological space, and let $B \subset Y$ with $Y - B$ countable. Suppose that for every integer $m \leq n$, the space Y^m has a base $\mathcal{W}_m$ which is closed under countable unions and has the property that $Y^m - W$ is countable whenever $W \in \mathcal{W}_m$ and $W \supset Y^m - B^m$. Then Y^n is Lindelöf.

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15. The CONTINUUM HYPOTHESIS states that there are no sets A such that $\aleph_0 < |A| < 2^{\aleph_0}$.

16. A space is said to be LINDELÖF if and only if every open cover has a countable subcover.

Proof: We will prove by induction that Y^m is Lindelöf for all $m \leq n$. The assertion is clearly true for $m = 0$ where Y^o is a one point set. Now assume that Y^{m-1} is Lindelöf and we will prove that Y^m is Lindelöf.

Let $\mathcal{U}$ be an open covering of Y^m. Then we must find a countable subcovering. Without loss of generality, we may assume that $\mathcal{U} \subset \mathcal{W}_m$. Now $Y^m - B^m$ is Lindelöf by our inductive hypothesis and Lemma 9.1. Therefore $\mathcal{U}$ has a countable subcollection $\mathcal{V}$ which covers $Y^m - B^m$. Let $W = \cup\mathcal{V}$. Then $W \in \mathcal{W}_m$ and $W \supset Y^m - B^m$, so $Y^m - W$ is countable. Let $\mathcal{V}_1$ be a countable subcollection of $\mathcal{U}$ which covers $Y^m - W$. Then $\mathcal{V} \cup \mathcal{V}_1$ is a countable subcover of $\mathcal{U}$ and so Y^m is Lindelöf.

Lemma 9.5. Let X be a T_1 - space, let A be a non - G_δ subset of X, and for each m let $\mathcal{U}_m$ be a collection of open subsets of X^m with $|\mathcal{U}_m| \leq \aleph_1$. Then there exists an uncountable subset B of $X - A$ such that if $Y = A \cup B$, then $Y^m - U$ is countable whenever $U \in \mathcal{U}_m$ with $U \supset Y^m - B^m$.

Proof: Let $\mathcal{U} = \bigcup_{m=1}^{\infty} \mathcal{U}_m$. Then $|\mathcal{U}| \leq \aleph_1$, so we can write $\mathcal{U} = \{U_\alpha | \alpha < \Omega\}$. By transfinite induction, we will construct a subset $B = \{y(\alpha) \mid \alpha < \Omega\}$ of $X - A$ with all $y(\alpha)$ distinct so that if $m \in N$, $U_\beta \in \mathcal{U}_m$, $U_\beta \supset Y^m - B^m$, and if $\alpha_1, \cdots, \alpha_m < \Omega$ with $\beta \leq \max\{\alpha_1, \cdots, \alpha_m\}$, then $(y(\alpha_1), \cdots, y(\alpha_m)) \in U_\beta$. This will complete the proof.

Let us suppose that $U^o = X$ and start the induction by letting $y(0)$ be any element of $X - A$. Now let $\alpha > 0$ and suppose that distinct y_β with $\beta < \alpha$ have been chosen so that our requirement is satisfied

whenever $\max\{\alpha_1,\cdots,\alpha_m\} < \alpha$. We must choose $y(\alpha)$ so that it is also satisfied whenever $\max\{\alpha_1,\cdots,\alpha_m\} = \alpha$, and so that $y(\alpha) \neq y(\beta)$ for all $\beta < \alpha$.

Let $B_\alpha = \{y(\beta) \mid \beta < \alpha\}$ and $Y_\alpha = A \cup B_\alpha$. For each integer m , let $\Gamma_m = \{\beta \leq \alpha \mid U_\beta \in \mathcal{U}_m , U_\beta \supset Y_\alpha^m - B_\alpha^m\}$. Then Γ_m is countable.

Now for each integer m , let ϕ_m be the set of all functions ϕ from $\{1,\cdots,m\}$ to $\{\beta \mid \beta \leq \alpha\}$ such that $\phi(i) = \alpha$ for at least one $i \leq m$. For each $\phi \in \phi_m$, define $g_\phi : X \to X^m$ by

$$(g_\phi(x))_i = \begin{cases} x & \text{if} \quad \phi(i) = \alpha \\ y(\phi(i)) & \text{if} \quad \phi(i) < \alpha \end{cases}$$

Now let $W_m = \cap\{g^{-1}(U_\beta) \mid \phi \in \phi_m , \beta \in \Gamma_m\}$, $W = \bigcap_{m=1}^{\infty} W_m - \{y(\beta) \mid \beta < \alpha\}$. Then W is a G_δ subset of X containing A , and hence contains an element of $X - A$ which we take to be $y(\alpha)$.

It remains to show that our requirements are now satisfied whenever $\max\{\alpha_1,\cdots,\alpha_m\} = \alpha$. So suppose that $\beta \leq \alpha$ and that $U_\beta \in \mathcal{U}_m$ and $U_\beta \supset Y^m - B^m$. Then $U_\beta \supset (Y_\alpha^m - B_\alpha^m)$, so $\beta \in \Gamma_m$. To show that $(y(\alpha_1),\cdots,y(\alpha_m)) \in U_\beta$, define $\phi \in \phi_m$ by $\phi(i) = \alpha_i$ for $1 \leq i \leq m$. Then $g_\phi(y(\alpha)) = (y(\alpha_1),\cdots,y(\alpha_m))$. But $y_\alpha \in W \subset W_m \subset g_\phi^{-1}(U_\beta)$, so $g_\phi(y(\alpha)) \in U_\beta$. Hence $(y(\alpha_1),\cdots,y(\alpha_m)) \in U_\beta$ and we are done.

The following theorem provides the main result with which we will construct our example. Note that the proof requires the continuum hypothesis.

Theorem 9.6. Let X be a T_1 space of weight $\leq 2^{\aleph_0}$, and A a countable, non G_δ subset of X. Then X has an uncountable subset $Y \supset A$ such that Y^n is Lindelöf for all integers n.

Proof: For each integer m, the space X^m has a base $\mathcal{U}_m$ of cardinality $2^{\aleph_0}$, and we may suppose that $\mathcal{U}_m$ is closed under countable unions. By the continuum hypothesis, we have that the cardinality of $\mathcal{U}_m \leq \aleph_1$. Now we apply Lemma 9.5 to find an uncountable subset B of $X - A$ such that if $Y = A \cup B$, then $Y^m - U$ is countable whenever $m \in N$, $U \in \mathcal{U}_m$, and $U \supset Y^m - B^m$. Let $\mathcal{W}_m = \{U \cap Y^m \mid U \in \mathcal{U}_m\}$. Then Y, B, and the $\mathcal{W}_m$ satisfy the assumptions of Lemma 9.4 for every integer n, so Y^n is Lindelöf for all $n \in N$.

Now let X be the space described in Example 1, and let A be the subset Q of rationals. Then X and A satisfy the hypothesis of Theorem 9.6, and so X has an uncountable subset Y containing A such that Y^n is Lindelof for every integer n. To show that $Y^{\aleph_0}$ is not normal, we shall show that $Y \times \underset{\sim}{P}$ is not normal and apply Theorem 9.3.

More generally, we can prove that $S \times \underset{\sim}{P}$ is not normal if S is any uncountable Lindelöf subspace of X containing Q. This follows because if S is an uncountable Lindelöf subspace of X containing Q, then every neighborhood of Q in S has a countable complement in S, so that Q is not a G_δ in S. However, we can also prove that $S \times \underset{\sim}{P}$ is not normal if S is any subspace of X containing Q as a non G_δ subset. The proof is the same as the proof that $Y \times \underset{\sim}{P}$ is not normal in Example 1. Hence $Y \times \underset{\sim}{P}$ is not normal and, by Theorem 9.3, $Y^{\aleph_0}$ is not normal.

Example 3. There exist semi-metrizable[17], regular spaces Y_1 and Y_2 such that $Y_1^{\aleph_o}$ and $Y_2^{\aleph_o}$ are both hereditarily Lindelöf[18], but $Y_1 \times Y_2$ is not normal.

As in the last example, we shall assume that the continuum hypothesis is true. Consider the space $X = R^2$. Let τ_1 be the topology on R^2 generated by the base consisting of all "horizontal bow-tie neighborhoods", and let τ_2 be the topology on R^2 generated by all "vertical bow-tie neighborhoods". Then it can be shown that (X,τ_1) and (X,τ_2) are homeomorphic, and that (X,τ_1) and (X,τ_2) are completely regular, semi-metrizable topologies on X .

We will not go into the detail of this example. However, Michael was able to prove that X has a subset Y containing Q^2 such that Y^n is Lindelöf with respect to both τ_1 and τ_2 for every integer n . Now letting $Y_1 = (Y,\tau_1)$ and $Y_2 = (Y,\tau_2)$, we have that Y_1 and Y_2 are semi-metrizable and regular. Furthermore, it is possible to show that $Y_1^{\aleph_o}$ and $Y_2^{\aleph_o}$ are hereditarily Lindelöf. However, $Y_1 \times Y_2$ is not normal. This follows because if we let $E = (y,\tau_1) \times (Y,\tau_2)$, then since $Q^2 \subset Y$ we have that $Q^2 \times Q^2$ is a countable dense subset of E. On the other hand $D = \{(x,x) | x \in Y\}$ is a closed discrete subset of E of cardinality $2^{\aleph_o}$. Hence E is not normal by a theorem due to F.B. Jones ([W], Lemma 15.2, p. 100).

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17. A space X is said to be SEMI-METRIZABLE if there exists a function $d : X \times X \to R$ such that for all $x,y \in X$, (a) $d(x,y) = 0$ if and only if $x = y$, and (b) $d(x,y) = d(y,x)$.

18. A space is said to be HEREDITARILY LINDELÖF if and only if every subspace is Lindelöf.

Example 4. (Assumes the continuum hypothesis.) For each integer n, there exists a regular space Y such that Y^n is hereditarily Lindelöf, but Y^{n+1} is not normal.

Let X stand for the set of reals R equipped with any T_1 topology of weight $\leq 2^{\chi_o}$. We will define $D_n = \{x \in X^n \mid \sum_{i=1}^{n} x_i = \sqrt{2}\}$ and a subset E of X^n will be called SIMPLE if $y, y' \in E$ with $y \neq y'$ implies that $y_i \neq y_i'$ for all $i \leq n$. Michael provides the following theorem which we will state without proof.

Theorem 9.7. For any integer n, X has a subspace Y such that Y^n is Lindelöf and such that $Y^{n+1} \cap D_{n+1}$ contains an uncountable simple set E.

Now consider the Soyenfrey line S. Clearly S satisfies all of the conditions of the above theorem, and so we can find Y and E as in the theorem such that Y^n is Lindelöf and E is an uncountable simple set contained in $Y^{n+1} \cap D_{n+1}$. However, every open subset of S^{ω}, where $\omega \leq \chi_o$, is an F_σ. Hence every open subset of Y^n is an F_σ, from which it follows that Y^n is hereditarily Lindelöf. (Since every open subset is an F_σ, we immediately have that every open or closed subset is Lindelof. Now suppose that B is any other subset and let $\{U_\alpha\}$ be an open cover of B. Then $U = \bigcup_\alpha U_\alpha$ is an open set containing B. But U is Lindelöf and so there exists a countable subcover $\{U_n\}$ which covers U and hence B. So B is Lindelöf.)

To show that Y^{n+1} is not normal, we observe that Y is separable and so Y^{n+1} is also separable. Since we have assumed that the

continuum hypothesis is true, the cardinality of E is $2^{\aleph_0}$. We will be done if we can show that E is discrete and closed in Y^{n+1}. But E is a subset of D_{n+1} and clearly D_{n+1} is closed and discrete in Y^{n+1}. So E is a closed and discrete subset of Y^{n+1}. Hence, by Jones' Lemma ([W], Lemma 15.2, p. 100), Y^{n+1} is not normal.

CHAPTER III

Other Properties in Cartesian Products

§10. Introduction. Having discussed the properties of paracompactness and normality in the previous two chapters, we now turn our attention to some other properties in product spaces, namely the Lindelöf property, countable compactness, pseudocompactness, and perfect normality. We will show by example that the product of spaces having the property P need not have that property, where P stands for one of the above properties. Having done this, we will be interested in determining conditions under which the product will have the desired property.

§11. The Lindelöf Property. Recall that a topological space is said to be LINDELÖF if every open cover has a countable subcover. Then a Lindelöf space is a space which is almost compact, and apparently, a countably compact, Lindelöf space is compact. It is well known that a continuous image of a Lindelöf space is again Lindelöf, and that a closed subspace of a Lindelöf space is Lindelöf ([W], p. 110). One of the more important facts concerning Lindelöf spaces is that every regular, Lindelöf space is normal ([W], p. 111). Finally, the Sorgenfrey line (§1.1) is an example of a Lindelöf space whose product with itself is not Lindelöf.

There are several results available for a Lindelöf product. The first result is due to K. Morita, and it expresses a necessary and sufficient condition on a space X such that the product space $X \times Y$ is Lindelöf for every separable metric space Y. A space is said to be

SEPARABLE if it contains a countable dense subset. Morita's result is based upon many of the definitions and results given in §2. We shall need an additional result.

Theorem 11.1. A space X is a separable metric space if and only if there exists a subspace S of the Cantor discontinuum D^{χ_0} and a closed continuous mapping g from S onto X such that $g^{-1}(x)$ is compact for each point $x \in X$.

Proof: Let Ω be a countably infinite set. Then any subspace of $N(\Omega)$ is a separable metric space of dimension ≤ 0, and hence is homeomorphic to a subspace of D^{χ_0}. Therefore, Theorem 11.1 is an immediate consequence of Theorem 2.11.

Theorem 11.2. The product $X \times Y$ is Lindelof and normal for every separable metric space Y if and only if X is a Lindelof, normal $P(2)$ - space.

Proof: $\longrightarrow$. That X is Lindelof is clear. Let S be any subspace of the Baire space $N(\Omega)$, where Ω is countably infinite. Then S is a separable metric space, and by assumption, $X \times S$ must be normal. Then, by Lemma 2.15, X must be a normal $P(m)$ - space where $m \geq 2$, and so X is a normal $P(2)$ - space.

$\longleftarrow$. Let X be a normal $P(2)$ - space with the Lindelöf property, and let S be any subspace of the Cantor discontinuum $C = D^{\chi_0}$ where $D = \{0,1\}$. Now let $\mathcal{M} = \{M_\lambda \mid \lambda \quad \Lambda\}$ be any open covering of $X \times S$. Then as a refinement of $\mathcal{M}$ we can find an open covering

(1) $$\{L(\epsilon_1,\cdots,\epsilon_i;\lambda) \times [V(\epsilon_1,\cdots,\epsilon_i) \cap S] \mid \epsilon_1,\cdots,\epsilon_i = 0,1;$$
$$i = 1,2,\cdots , \lambda \in \Lambda\}$$

of $X \times S$ such that $L(\epsilon_1,\cdots,\epsilon_i;\lambda) \times [V(\epsilon_1,\cdots,\epsilon_i) \cap S] \subset M_\lambda$. Let us put $G(\epsilon_1,\cdots,\epsilon_i;\lambda) = \bigcup_{j=1}^{i} L(\epsilon_1,\cdots,\epsilon_j;\lambda)$. Then

(2) $$G(\epsilon_1,\cdots,\epsilon_i;\lambda) \times [V(\epsilon_1,\cdots,\epsilon_i) \cap S] \subset M_\lambda$$

since $V(\epsilon_1,\cdots,\epsilon_i) \subset V(\epsilon_1,\cdots,\epsilon_j)$ for $j \leq i$. Now let us put

(3) $$G(\epsilon_1,\cdots,\epsilon_i) = \cup\{G(\epsilon_1,\cdots,\epsilon_i;\lambda) \mid \lambda \in \Lambda\} .$$

Then it follows that $G(\epsilon_1,\cdots,\epsilon_i) \subset G(\epsilon_1,\cdots,\epsilon_i,\epsilon_{i+1})$ for $\epsilon_1,\cdots,\epsilon_{i+1} = 0,1$, and

(4) $$X = \bigcup_{i=1}^{\infty} G(\epsilon_1,\cdots,\epsilon_i) \quad \text{for} \quad (\epsilon_1,\epsilon_2,\cdots) \in S ,$$

since (1) is a covering of $X \times S$.

From the assumption that X is a normal $P(2)$ - space and from Lemma 2.1, it follows that there exists a family

$$\{H(\epsilon_1,\cdots,\epsilon_i) \mid \epsilon_1,\cdots,\epsilon_i = 0,1 ; i = 1,2,\cdots\}$$

of open F_σ - subsets of X such that $H(\epsilon_1,\cdots,\epsilon_i) \subset G(\epsilon_1,\cdots,\epsilon_i)$ and

(5) $$X = \bigcup_{i=1}^{\infty} H(\epsilon_1,\cdots,\epsilon_i) \quad \text{for} \quad (\epsilon_1,\epsilon_2,\cdots) \in S .$$

Since each subspace $H(\epsilon_1,\cdots,\epsilon_i)$ is an F_σ subset of the Lindelöf space X, $H(\epsilon_1,\cdots,\epsilon_i)$ has the Lindelöf property. Hence there exists a countable subset $\Lambda(\epsilon_1,\cdots,\epsilon_i)$ of Λ such that

$$\{G(\epsilon_1,\cdots,\epsilon_i;\lambda) \cap H(\epsilon_1,\cdots,\epsilon_i) \mid \lambda \in \Lambda(\epsilon_1,\cdots,\epsilon_i)\}$$

is a countable covering of $H(\epsilon_1,\cdots,\epsilon_i)$. Then the family $\{[G(\epsilon_1,\cdots,\epsilon_i;\lambda) \cap H(\epsilon_1,\cdots,\epsilon_i)] \times [V(\epsilon_1,\cdots,\epsilon_i) \cap S] \mid \lambda \in \Lambda(\epsilon_1,\cdots,\epsilon_i)$, $\epsilon_1,\cdots,\epsilon_i = 0,1$; $i = 1,2,\cdots\}$ is a countable open covering of $X\times S$ in view of (3), (4), and (5). Moreover, by (2), it is a refinement of $\mathcal{M}$. Therefore, $X\times S$ has the Lindelöf property.

Now let Y be any separable metric space. Then, by Theorem 11.1, there exists a subspace S of $C = D^{\chi_0}$ and a continuous mapping of g from S onto Y. Hence there exists a continuous mapping from $X\times S$ onto $X\times Y$. Since $X\times S$ is Lindelöf, it follows that $X\times Y$ has the Lindelöf property.

Morita's theorem provides us with the best result for a Lindelöf product $X\times Y$ where Y is any separable metric space. For larger products, we have a result due to E. Michael. However, before looking at this, we will examine another of Michael's counterexamples. In this one, he is able to show the existence, for every integer n, of a regular space X such that X^n is Lindelöf, X^{n+1} is paracompact, but X^{n+1} is not Lindelöf.

Example 5. Let X be the space in Example (1) of §9. Recalling the hypothesis which led to Theorem 9.7 in Example (4) of §9, we see that the space X satisfies these conditions. Therefore we can pick the sets Y

and E as in that example and Y^n will be Lindelöf. We must now show that Y^{n+1} is not Lindelöf, and to accomplish this we will show that the uncountable subset E of Y^{n+1} is closed and discrete. Now D_{n+1} is surely closed in X^{n+1}. Therefore we need only show that if $x \in D_{n+1}$, then x has a neighborhood in X^{n+1} containing at most one element of E. But clearly $D_{n+1} \cap Q^{n+1} = \phi$, so x_i is irrational for some $i \leq n+1$, and since E is simple, $\{x' \in X^{n+1} \mid x'_i = x_i\}$ is a neighborhood of x containing at most one element of E. Hence E is closed and discrete, and so Y^{n+1} is not Lindelöf. That X^{n+1} is paracompact follows from Theorem 9.2.

In addition to the above counterexample, Michael is able to provide a condition which guarantees that a countable product of topological spaces will be Lindelöf.

<u>Theorem 11.3</u>. If for all integers n, $\prod_{i=1}^{n} X_i$ is a Lindelöf space in which every open subset is an F_σ, then $\prod_{i=1}^{\infty} X_i$ is Lindelöf.

<u>Proof</u>: Let $\mathcal{U}$ be an open cover of $\prod_{i=1}^{\infty} X_i$. Then for each $U \in \mathcal{U}$, $U = \bigcup_{n=1}^{\infty} (U(n) \times \prod_{i=n+1}^{\infty} X_i)$ where $U(n)$ is an open set in $\prod_{i=1}^{n} X_i$. Let $V(n) = \cup\{U(n) \mid U \in \mathcal{U}\}$. Then $\{U(n) \mid U \in \mathcal{U}\}$ is an open cover of $V(n)$. But since every open subset of $\prod_{i=1}^{n} X_i$ is an F_σ, we get that $\prod_{i=1}^{n} X_i$ is hereditarily Lindelöf, and so $V(n)$ is Lindelöf. Hence there exists a countable subcollection $\mathcal{U}_n$ of $\mathcal{U}$ such that $V(n) = \cup\{U(n) \mid U \in \mathcal{U}_n\}$. Then $\{U(n) \times \prod_{i=n+1}^{\infty} X_i \mid U \in \mathcal{U}_n, n \in N\}$ is a countable subcollection from

$\mathcal{U}$ which covers $\prod_{i=1}^{\infty} X_i$. Hence $\prod_{i=1}^{\infty} X_i$ is Lindelöf.

§12. Countable Compactness. Recall that a space X is said to be COUNTABLY COMPACT if every countable open cover of X has a finite subcover. It is readily shown that this is equivalent to every countably infinite subset G of X having a cluster point. Obviously every compact space is countably compact, and so the property of countable compactness is a property slightly weaker than compactness. The following example shows that the product of two countably compact spaces need not be countably compact.

Example 6. We will construct a subspace G of N such that G is countably compact, but $G\times G$ is not countably compact. First, define a homeomorphism ψ of βN onto itself as follows. We note that βN is the union of two disjoint copies of itself, βN_1 and βN_2 . If we let τ be any homeomorphism of βN_1 onto βN_2 , then we define ψ to agree with τ on βN_1 and with τ^{-1} on βN_2 . Then we have that ψ has no fixed point and $\psi\cdot\psi$ is the identity on βN .

We will define the subspace G of βN inductively. Let ζ denote the family of all countably infinite subsets of βN . Since $|\beta N| = 2^c$, it follows that $|\zeta| = (2^c)^{\aleph_0} = 2^c$. Let $<$ be a well ordering of ζ according to the smallest ordinal of cardinal 2^c . Consider any $S \in \zeta$, and suppose that for each $E < S$, we have chosen a cluster point p_E of E , distinct from $\psi(p_{E'})$ for all $E' < S$. Now $|\overline{S}| = 2^c$ and so $|\overline{S}-S| = 2^c$. Since the set of all predecessors of S is of smaller cardinal, we can select p_S in $\overline{S}-S$ so as to differ from $\psi(p_E)$ for

all $E < S$. We now define $G = N \cup \{p_S \mid S \in \zeta\}$. By construction, every countably infinite subset of G has a cluster point in G , and so G is countably compact.

To show that $G \times G$ is not countably compact, we need only show that there exists an unbounded continuous function on $G \times G$ ([W], Theorem 17.13, p. 123). To show this, it is sufficient to prove that there exists an infinite, discrete set that is open and closed. Consider the infinite set $D = \{(n,\psi(n)) \mid n \in N\}$. Since ψ carries N into N , each point of D is isolated, and hence D is open and discrete. On the other hand, if $p \notin N$, then G by construction does not contain both p and $\psi(p)$. Therefore, D is the intersection of $G \times G$ with the subset $\{(p,\psi(p)) \mid p \in \beta N\}$ of $\beta N \times \beta N$. But the latter set is the graph of a continuous mapping and hence is closed. Therefore D is closed in $G \times G$. (Those wishing to see the details of this example should refer to [GJ], p. 135).

Z. Frolik is responsible for our first result concerning countably compact products. He makes use of the easily demonstrated fact that X is countably compact if and only if every countable family of closed subsets of X with the finite intersection property has a non-empty intersection. Before we state and prove Frolik's theorem, a couple of lemmas are needed.

<u>Lemma 12.1</u>. Let f be a closed map from a space P into a space Q . If Q is countably compact and if $f^{-1}(y)$ is countably compact for each $y \in Q$, then P is countably compact.

Proof: Let $\{F\}$ be a countable family of closed subsets of P with the finite intersection property. Without loss of generality, we may assume that if F_1 and F_2 belong to $\{F\}$, then their intersection $F_1 \cap F_2$ also belongs to $\{F\}$. Now choose a point $y \in \cap\{f(F)\}$. By assumption, the space $E = f^{-1}(y)$ is countably compact. The family $\{F \cap E\}$ is a countable collection of closed subsets of E and has the finite intersection property. Hence $\cap\{E \cap F\} \neq \phi$ and so $\cap\{F\} \neq \phi$. Therefore P is countably compact.

Lemma 12.2. Let K be a compact space and P be any space. Then the projection π_p of the product $P \times K$ onto P is a closed map.

Proof: Let F be a closed subset of $P \times K$ and $x \in (P - \pi_p(F))$. For each $y \in K$ there exist open sets $U(y) \subset P$ and $V(y) \subset K$ such that $x \in U(y)$, $y \in V(y)$, $[U(y) \times V(y)] \cap F = \phi$. Choose a finite subset Y of K so that $\cup\{V(y) \mid y \in Y\} = K$. Then the intersection $U = \cap\{U(y) \mid y \in Y\}$ is a neighborhood of x and $U \cap \pi_p(F) = \phi$. Hence $\pi_p(F)$ is closed and so π_p is a closed map.

Frolik's result is an immediate consequence of these two lemmas.

Theorem 12.3. If P is countably compact and if K is compact, then the product $P \times K$ is countably compact.

Proof: By Lemma 12.2, the projection π_p is a closed map from $P \times K$ into P, and we are given that P is countably compact. Therefore, if we can show that $\pi_p^{-1}(y)$ is countably compact in $P \times K$ for every $y \in P$, then we're done by Lemma 12.1. But $\pi_p^{-1}(y)$ is homeomorphic to K which is

compact, and hence countably compact. Therefore $\pi_p^{-1}(y)$ is countably compact, which completes the proof.

This is Frolik's only positive result on the question of a countably compact product. On the other hand, he is able to characterize those spaces whose product with a countably compact space is NOT countably compact.

Let $\mathcal{F}$ denote the class of all completely regular spaces X whose product with every countably compact, completely regular space Y is countably compact. The class $\mathcal{F}$ has the following properties:

<u>Lemma 12.4</u>. Let $X_1, X_2 \in \mathcal{F}$.

(a) If F is a closed subspace of X_1, then $F \in \mathcal{F}$.

(b) $X_1 \times X_2 \in \mathcal{F}$.

<u>Proof</u>: (a) Since F is a closed subspace of X, it follows that $F \times Y$ is a closed subspace of $X \times Y$. Since closed subspaces of countably compact spaces are again countably compact, it follows that $F \times Y$ is countably compact. Hence $F \in \mathcal{F}$.

(b) $X_2 \in \mathcal{F}$ implies that $X_2 \times Y$ is countably compact for every countably compact, completely regular space Y. $X_2 \times Y$ must also be completely regular, and since $X_1 \in \mathcal{F}$, it follows that $X_1 \times (X_2 \times Y)$ is countably compact. Hence $(X_1 \times X_2) \times Y$ is countably compact and so $X_1 \times X_2 \in \mathcal{F}$.

Frolik is now able to characterize those spaces P which do not belong to the class $\mathcal{F}$.

Theorem 12.5. Let P be a completely regular space. Then $P \notin F$ if and only if P satisfies the following condition: there exists an infinite discrete subset N of P such that for every compactification K of P, there exists a subset S of $K-P$ such that the subset $N \cup S$ of K is countably compact.

Proof: $\longleftarrow$. Suppose that the condition is satisfied. Then there exist sets $N \subset P$ and $S \subset K-P$ where K is some compactification of P such that $N \cup S$ is countably compact. Now if $P \in F$, then $P\times(N\cup S)$ must be countably compact. Consider the infinite discrete closed subset $\{(n,n) \mid n \in N\}$ of $P\times(N\cup S)$. Clearly this set is countably compact, and since a closed subset of a countably compact space is countably compact, it follows that $P\times(N\cup S)$ is not countably compact. Hence $P \notin F$.

$\longrightarrow$. Let P be a completely regular space such that $P \notin F$. Then there exists a countably compact, completely regular space Q such that $P\times Q$ is not countably compact. Two cases can arise:

(1) if P is not countably compact, then it contains an infinite closed discrete subset N. Let us put $S = \overline{N}^K - N$ where $\overline{N}^K$ denotes the closure of N is some compactification K of P. The space $N \cup S$ is compact and hence satisfies the condition.

(2) suppose P is countably compact. $P\times Q$ is not countably compact and so there exists an infinite closed discrete subset N' of $P\times Q$. Denote by π_P and π_Q the projections of $P\times Q$ onto P and Q respectively. Then π_P and π_Q are open and continuous. Since the spaces P and Q are countably compact, the sets $\pi_P^{-1}(x) \cap N'$ and

$\pi_Q^{-1}(y) \cap N'$ are finite for each $x \in P$ and $y \in Q$. Then it follows that for some infinite subset N'' of N', the sets $\pi_P^{-1}(x) \cap N''$ and $\pi_Q^{-1}(y) \cap N''$ contain at most one point. That is, $(x,y) \in N''$, $(x',y') \in N''$, $(x,y) \neq (x',y')$ implies that $x \neq x'$, $y \neq y'$. Since every infinite Hausdorff space contains an infinite discrete subset, we can choose an infinite subset N of N'' such that the sets $N_1 = \pi_P(N)$ and $N_2 = \pi_Q(N)$ are discrete. Thus N represents a one-to-one mapping from N_2 onto N_1. For each $y \in \overline{N}_2 - N_2$, denote by $\mathcal{U}(y)$ the family of all neighborhoods of $y \in Q$.

Now let K be any compactification of P. For each $y \in \overline{N}_2 - N_2$, put $\alpha(y) = \cap\{\overline{N(U \cap N_2)}^K \mid U \in \alpha(y)\}$. Since the space K is compact, the sets $\alpha(y)$ are compact and non-empty. Furthermore, the sets $\alpha(y)$ are disjoint from P. For suppose that there exists a point $x \in \alpha(y) \cap P$. We assert that the point (x,y) is an accumulation point of the set N, which is impossible. Let U be a neighborhood of the point $x \in P$ and let V be a neighborhood of $y \in Q$. According to the definition of $\alpha(y)$, the set $U \cap N(V \cap N_2)$ is infinite and clearly $N \cap (U\ V) \supset \{(x, N^{-1}(x)) \mid x \in U \cap N(V \cap N_2)\}$.

Now let us put $S = \cup\{\alpha(y) \mid y \in \overline{N}_2 - N_2\}$. It remains to prove that the space $N_1 \cup S = R$ is countably compact. First let N' be an infinite subset of N_1. The set $N^{-1}(N')$ has an accumulation point $y \in \overline{N}_2 - N_2$. It is easy to see that $\alpha(y) \cap \overline{N'}^K \neq \phi$. Indeed the family $\{N' \cup N(U \cap N_2) \mid U \in \mathcal{U}(y)\}$ has the finite intersection property.

Now let N' be an infinite discrete subset of S. If for some $y \in \overline{N}_2 - N_2$ the set $N' \cap \alpha(y)$ is infinite, then N' has an accumulation point in $\alpha(y)$ since $\alpha(y)$ is a compact space. In the other case, the sets $\alpha(y) \cap N'$ are finite, and without loss of generality we may assume that they contain at most one point. For each $x \in N'$ choose a point $\beta(x) \in \overline{N}_2 - N_2$ such that $x \in \alpha(\beta(x))$. By our assumption, the function β is one-to-one. Then the set $\beta(N')$ must be infinite. Let y be an accumulation point of $\beta(N')$ in $\overline{N}_2 - N_2$. We shall prove that $\alpha(y) \cap \overline{N'}^K \neq \phi$. Let $\mathcal{B} = \{\overline{N(U \cap N_2)}^K \mid U \in \mathcal{U}(y)\}$. By construction, we know $\alpha(y) = \cap\{B \mid B \in \mathcal{B}\}$. To prove the last statement, it is sufficient to show that $B \in \mathcal{B}$ implies that $B \cap \overline{N'}^K \neq \phi$. Suppose that $B = \overline{N(U \cap N_2)}^K$, where $U \in \mathcal{U}(y)$. The point y is an accumulation point of the set $\beta(N')$ and therefore we may choose an interior point y' of U belonging to $\beta(N')$. Hence $U \in \mathcal{U}(y')$ and $\alpha(y') \subset B$, and so $B \cap N' \neq \phi$. Therefore, if y is an accumulation point of $\beta(N')$ in $\overline{N}_2 - N_2$, then $\alpha(y) \cap \overline{N'}^K \neq \phi$. An analogous assertion holds for every infinite subset N'. Choose an accumulation point y of $\beta(N')$. It is easy to conclude that $\alpha(y) \cap N' = \phi$. Indeed $\alpha(y) \subset \overline{N' - (x)}^K$ for each $x \in N'$. It follows that every point of $\alpha(y)$ is an accumulation point of N', and the proof is complete.

Note that Frolik's study was restricted to completely regular spaces. Noble is also able to provide a sufficient condition on a space X in order that the product space $X \times Y$ is countably compact for every countably compact space Y. Then, without assuming complete regularity, he is able to characterize those spaces which satisfy the condition.

Denote by ζ the class of all spaces X such that the product space $X \times Y$ is countably compact for every countably compact space Y. A space X is called a k-SPACE if each subset of X which meets every compact subset in a relatively closed set is itself closed. Note that for each space X, there is a unique k-space, say kX having the same underlying set and the same compact subsets as X. The space kX is formed by adjoining to the topology on X all those subsets whose complements meet each compact set in a relatively closed set.

Let ζ^* denote the class of all spaces X such that each infinite subset of X meets some compact subset of X in an infinite set. We will show that $\zeta^* \subset \zeta$.

<u>Theorem 12.6</u>. If $X \in \zeta^*$, then $X \times Y$ is countably compact for every countablyccompact space Y.

<u>Proof</u>: Suppose that $X \times Y$ contains a countably infinite closed discrete subset $\{(x_n, y_n)\}$. Let K be a compact subset of X such that $K \cap \{x_n\} = \{x_{n_i}\}$ is infinite. Note that $\{x_n\}$ must be infinite since Y is countably compact. Then since $K \times Y$ is countably compact, $\{(x_{n_i}, x_{n_i})\}$ must have a cluster point. Hence $X \times Y$ is countably compact.

Noble now characterizes those spaces X which are in the class ζ^*.

<u>Theorem 12.7</u>. $X \in \zeta^*$ if and only if kX is countably compact. Thus each countably compact k-space is contained in ζ^*.

Proof: $\longleftarrow$. Suppose that kX is countably compact. Then any countable subset which meets each compact set in only finitely many points must be closed, and hence finite since each countable closed set is compact. Therefore $X \in \zeta^*$.

$\longrightarrow$. Suppose kX contains an infinite closed discrete subspace D . Then $D \cap K$ must be finite for each compact subset K of X .

§13. Pseudocompactness. A pseudocompact space, like a countably compact space, is a space which is almost compact. A space X is said to be PSEUDOCOMPACT if every continuous, real valued function on X is bounded. There isn't a great deal of difference between pseudocompactness and countably compactness. In fact, every countably compact space is pseudocompact and in normal spaces, they are equivalent. Obviously, every compact space is pseudocompact, and every closed subspace of a pseudocompact space is pseudocompact. But the product of two pseudocompact spaces need not be pseudocompact. Example 6 of §12 provides the counterexample.

The main results for pseudocompact products have been obtained by I. Glicksberg, Z. Frolik, and N. Noble. Their results will be examined individually and in chronological order.

Glicksberg discusses pseudocompactness of a space with respect to its convergence properties. He gives the following characterization of pseudocompactness.

Theorem 13.1. A space X is pseudocompact if and only if every sequence of non-empty open sets in X has a cluster point.

Proof: $\longrightarrow$. A sequence of non-empty open sets yields a collection of sets which is either locally finite or not. If the collection is locally finite, some point lies in infinitely many sets of the sequence and is a cluster point. In the second case, there must be some point which prohibits local finiteness. This point is clearly the required cluster point.

$\longleftarrow$. Since every sequence of non-empty open sets in X has a cluster point, it follows that every locally finite collection of open sets is finite. That X is pseudocompact follows immediately.

Using this result, Glicksberg is able to prove the following theorem concerning infinite pseudocompact products.

Theorem 13.2. Any product of spaces is pseudocompact if and only if each countable subproduct is pseudocompact.

Proof: $\longrightarrow$. Clearly each factor space and each partial product must be pseudocompact since continuous functions on these spaces may be considered as continuous functions on the full product.

$\longleftarrow$. It suffices to show that every sequence of non-empty canonical neighborhoods in ΠX_α has a cluster point. Since each neighborhood places retrictions on only finitely many factor spaces, only countably many spaces are involved, say $X_{\alpha_1}, X_{\alpha_2}, \cdots$. Choosing any cluster point of the projection of our sequence into $\prod_i X_i$, we can, by arbitrary choice of all other coordinates, clearly extend this to a cluster point of the original sequence.

Glicksberg was also able to provide an answer to the following question: under what conditions on X and Y will the product $X \times Y$ be pseudocompact. When X and Y are infinite, completely regular spaces, the following theorem provides an answer.

Theorem 13.3. Let X and Y be infinite, completely regular spaces. Then $X \times Y$ is pseudocompact if and only if $\beta(X \times Y) = \beta X \times \beta Y$.

Glicksberg proves this theorem in a more general form, namely for arbitrary products. The proof of the above theorem will not be given now since it appears later as part of a theorem due to Frolik (Theorem 13.6).

Frolik attempts to find the conditions under which a product $X \times Y$ will be pseudocompact if X and Y are pseudocompact. Using Glicksberg's result concerning the Stone-Cech compactification of the product and by adding a third equivalent condition, he is able to simplify Glicksberg's proof considerably. Before stating Frolik's theorem, we must establish two lemmas.

Lemma 13.4. Let X and Y be completely regular infinite spaces. If the product $X \times Y$ is not pseudocompact, then there exists a locally finite sequence $\{U_n \times V_n\}$ of non-empty canonical open subsets of $X \times Y$ such that the sequences $\{U_n\}$ and $\{V_n\}$ are disjoint.

Proof: Suppose that one of the spaces, say X , is not pseudocompact. Then there exists a locally finite disjoint sequence $\{U_n\}$ of non-empty open subsets of X . Since Y is an infinite Hausdorff space, we can select a disjoint sequence $\{V_n\}$ of non-empty subsets of Y . Then the sequence $\{U_n \times V_n\}$ has the desired properties.

Now suppose that X and Y are pseudocompact spaces. Let $\{U'_n \times V'_n \mid n \in N\}$ be a locally finite sequence of non-empty open subsets of $X \times Y$. Suppose that $\{U'_n \times V'_n \mid n \in N'\}$ is a subsequence of the given sequence. We will show that for each $x \in X$, there exists an open neighborhood U of x such that $U \cap U'_n = \phi$ for an infinite number of $n \in N'$. For suppose the contrary. Then select a cluster point y of $\{V'_n \mid n \in N'\}$ and it readily follows that (x,y) is a cluster point of $\{U'_n \times V'_n \mid n \in N'\}$ which is a contradiction.

Therefore we can choose by induction a sequence $n_1, n_2 \cdots \in N$ and open non-empty sets $U_{n_i} \subset U'_{n_i}$ such that the sequence $\{U_{n_i}\}$ is disjoint. Applying the same argument to the sequence $\{U_{n_i} \times V'_{n_i}\}$, we obtain a subsequence $\{n_{i_k}\}$ of $\{n_i\}$ and open non-empty sets $V_{n_{i_k}} \subset V'_{n_{i_k}}$ such that the sequence $\{V_{n_{i_k}}\}$ is disjoint. The sequence $\{U_{n_{i_k}} \times V_{n_{i_k}}\}$ has the desired properties.

<u>Lemma 13.5</u>. Let f be a continuous function on a completely regular, pseudocompact space $X \times Y$. If K is a compactification of Y such that every function $f(x, \ )$ has a continuous extension to K, then f has a continuous extension to $X \times K$.

<u>Proof</u>: We can extend continuously every function $f(x, \ )$ on $\{x\} \times K$. In this manner, we obtain a function f^* on $X \times K$. We shall prove that f^* is continuous. Given $(x_o, y_o) \in X \times K$ and $\epsilon > 0$, we must find a neighborhood $W = U \times V$ of (x_o, y_o) such that $|f^*(x_o, y_o) - f^*(x,y)| < \epsilon$ for each $(x,y) \in W$. Choose an open neighborhood V of $y_o \in K$ such

that $|f^*(x_o,y) - f^*(x_o,y_o)| < \epsilon$ for each $y \in V$. Now the space $\overline{(V\cap Y)}^Y$ is pseudocompact and so we may choose an open neighborhood U of X_o such that for each $x \in U$ we have

$$\inf_{y\in V\cap Y} f(x,y) > f^*(x_o,y_o) - 2\epsilon$$

and

$$\sup_{y\in V\cap Y} f(x,y) < f^*(x_o,y_o) + 2\epsilon \quad .$$

Then it follows that $|f^*(x,y) - f^*(x_o,y_o)| < 2\epsilon$ for each $(x,y) \in U\times V$. Hence f^* is continuous.

Frolik's result on pseudocompactness in products is contained in the following theorem.

<u>Theorem 13.6</u>. Let X and Y be infinite, completely regular spaces. Then the following are equivalent.

(a) $X\times Y$ is pseudocompact.

(b) $\beta(X\times Y) = \beta X\times\beta Y$ (that is, every bounded, continuous function on $X\times Y$ has a continuous extension to $\beta X\times\beta Y$).

(c) if f is a bounded, continuous function on $X\times Y$, then for every $\epsilon > 0$, there exists a finite cover $\mathcal{U} = \{A_1,\cdots,A_n\}$ of $X\times Y$ consisting of canonical open sets A_i on each of which f varies $< \epsilon$.

<u>Proof</u>: (a) $\rightarrow$ (b). Let f be a continuous function on $X\times Y$. By Lemma 13.5, there exists a continuous extension of f to $X\times\beta Y$. By the same

lemma, this extension has a continuous extension to $\beta X\times\beta Y$.

(b) → (c) . Let f be a bounded, continuous function on $X\times Y$. Denote by f^* the continuous extension of f to $\beta X\times\beta Y$. Let $\mathcal{U}_1$ denote the family of all canonical open subsets of $\beta X\times\beta Y$ on which f^* varies $< \epsilon$. Then $\mathcal{U}_1$ is an open cover of the compact space $\beta X\times\beta Y$. Hence some finite subfamily $\mathcal{U}_2$ of $\mathcal{U}_1$ also covers $\beta X\times\beta Y$. The family $\mathcal{U} = \{U\cap(X\times Y) \mid U \in \mathcal{U}_2\}$ has the desired properties.

(c) → (a) . Suppose that $X\times Y$ is not pseudocompact. Then, by Lemma 13.4, there exists a locally finite sequence $\{W_n\} = \{U_n\times V_n\}$ of non-empty canonical open subsets of $X\times Y$ such that the sequences $\{U_n\}$ and $\{V_n\}$ are disjoint. Choose points $z_n \in W_n$ and continuous functions $f_n \leq 1$ such that $f_n(z_n) = 1$ and $f_n(z) = 0$ for each $z \notin W_n$. Then $f = \sum_{n=1}^{\infty} f_n$ is a bounded continuous function on $X\times Y$. If $A = A_1 \times A_2$ is a subset of $X\times Y$ containing two points z_n and z_k with $n \neq k$, then f varies ≥ 1 on A . Indeed, if $z_i = (x_i,y_i)$, $i = 1,2,\cdots$, then the point $(x_n,y_k) \in A$ and $f(x_n,y_k) = 0$. Hence $f(z_n) - f(x_n,y_k) = 1$. Hence condition (c) is not satisfied and we have a contradiction.

Let us denote by $\mathcal{D}$ the class of all completely regular spaces X such that $X\times Y$ is pseudocompact for every pseudocompact regular space Y . Frolik was also interested in the kinds of spaces which belong to the class $\mathcal{D}$. The following properties concerning the class $\mathcal{D}$ are evident:

(a) If $X \in \mathcal{D}$ and Y is a completely regular space which is a continuous image of X , then $Y \in \mathcal{D}$.

(b) $X, Y \in \mathcal{D}$ if and only if $X \times Y \in \mathcal{D}$.

(c) If F is a regularly closed subspace of $X \in \mathcal{D}$ (that is, $F = Cl_X(Int_X F)$) then $F \in \mathcal{D}$.

The next two theorems provide a great many examples of spaces which belong to the class $\mathcal{D}$.

Theorem 13.7. Compact spaces belong to $\mathcal{D}$.

Proof: Let X be a pseudocompact space and let K be a compact space. To prove that $X \times K$ is pseudocompact, it is sufficient to show that every bounded continuous function f on $X \times K$ assumes its lower bound. X is pseudocompact and hence assumes its lower bound at a point x . The function $f(x, \ )$ on $\{x\} \times K$ assumes its lower bound at a point y . Evidently f assumes its lower bound at (x,y) .

Theorem 13.8. A space X belongs to $\mathcal{D}$ if it satisfies the following condition: if $\mathcal{U}$ is an infinite disjoint family of non-empty open subsets of X , then for some compact subset K of X , the intersection $K \cap A$ is non-empty for an infinite number of sets $A \in \mathcal{U}$.

Proof: Suppose that $X \notin \mathcal{D}$. Then for some pseudocompact completely regular space Y , the product $X \times Y$ is not a pseudocompact space. By Lemma 13.4, there exists a locally finite sequence $\{U_n \times V_n\}$ of non-empty open subsets of $X \times Y$ such that the sequence $\{U_n\}$ is disjoint. By the given condition, there is a compact space K meeting an infinite number of sets U_n . Consider the space $X \times Y$. By Theorem 13.7, $K \times Y$ is pseudocom-

compact. But $\{(U_n \times V_n) \cap (K \times Y) \mid U_n \cap K \neq \phi\}$ is a locally finite sequence of non-empty open subsets of $K \times Y$, which is a contradiction. Hence $X \in \mathcal{D}$.

Finally, Frolik is able to characterize those spaces which belong to the class $\mathcal{D}$.

Theorem 13.9. A completely regular space X belongs to the class $\mathcal{D}$ if and only if it satisfies the following condition: if $\mathcal{U}$ is an infinite disjoint family of non-empty open subsets of X, then there exists a disjoint sequence $\{U_n\}$ in $\mathcal{U}$ such that for every filter $\mathcal{N}$ of infinite subsets of $N = \{n\}$, we have $\bigcap_{N_1 \in \mathcal{N}} \overline{\bigcup_{n \in N_1} U_n} \neq \phi$.

Proof: $\longleftarrow$. Suppose that the condition holds and let Y be a pseudocompact space, and suppose that $X \times Y$ is not pseudocompact. The spaces X and Y are infinite and by Lemma 13.4 there exists a sequence $\{U_n \times V_n\}$ of non-empty open subsets of $X \times Y$ such that the sequence $\{U_n\}$ is disjoint. By the above condition, there exists a subsequence $\{U_{n'}\}$ of $\{U_n\}$ such that $\bigcap_{N' \in \mathcal{N}} \overline{\bigcup_{n' \in N'} U_{n'}} \neq \phi$ for every filter $\mathcal{N}$ in the set N of all integers n. Consider the locally finite sequence $\{U_n \times V_n \mid n \in N\}$. Let y be a cluster point of $\{V_n \mid n \in N\}$. Let $\mathcal{B}$ be the family of all neighborhoods of the point y. For each $B \in \mathcal{B}$, put $N(B) = \{n \mid n \in N, B \cap V_n = \phi\}$. Then $\mathcal{N} = \{N(B) \mid B \in \ \}$ is a filter in N and by our assumption, we may choose an $x \in \bigcap_{N_1 \in \mathcal{N}} \overline{\bigcup_{n \in N_1} U_n}$. Evidently (x,y) is a cluster point of the sequence $\{U_n \times V_n \mid n \in N\}$ which gives us a contradiction. Hence $X \times Y$ is pseudocompact and $X \in \mathcal{D}$.

$\longrightarrow$. Suppose that X is a completely regular space which does not satisfy the given condition. Then there exists a countably infinite disjoint family $\{U_n \mid n \in N\}$ of non-empty open subsets of X such that if N_o is any infinite subset of N, then $\bigcap_{N_1 \in \mathcal{N}} \bigcup_{n \in N_1} U_n = \phi$ for some filter $\mathcal{N}$ in N_o. Selecting points z_n in U_n and denoting the set of all these z_n by Z, consider the space $Y = \overline{Z}^{\beta X} - (X-Z)$. By our assumption, the space Y is pseudocompact since every infinite subset Z_o of Z has an accumulation point in Y. If we denote by N' the set of all n's such that $z_n \in Z$, then there exists a filter $\mathcal{N}$ in N' such that $\bigcap_{N_1 \in \mathcal{N}} \overline{\bigcup_{n \in N_1} U_n} = \phi$. But every point of the non-empty set $\bigcap_{N_1 \in \mathcal{N}} \overline{\bigcup_{n \in N_1} (U_n \cap Z)}^{\beta X}$ is an accumulation point of Z in Y. We will show that $X \times Y$ is not pseudocompact. First note that the one point sets $\{z_n\}$ are open in Y. Consider the family $\mathcal{U} = \{U_n \times \{z_n\} \mid n \in N\}$ of non-empty subsets of $X \times Y$. We will show that $\mathcal{U}$ is locally finite. Let z be a cluster point of Z in Y. Consider the family of all intersections $Z \cap W$ where W is a neighborhood of z. Then, by our construction, $\bigcap_{Z_1 \in \mathcal{N}} \overline{\bigcup_{z_n \in Z_1} U_n}^X = \phi$ and hence for each $x \in X$ we may choose a neighborhood U of x and a set Z' in $\mathcal{N}$ such that $\overline{\bigcup_{z_n \in Z'} U_n} \cap U = \phi$. Selecting a neighborhood V of z in Y such that $Z \cap V = Z'$, it is easily seen that $U \times V$ is a neighborhood of (x,z) meeting no set $U_n \times \{z_n\}$, $n \in N$. Hence $X \times Y$ is not pseudocompact and so $X \notin \mathcal{D}$ which is a contradiction. Therefore the condition must hold.

Noble's work is quite similar to Frolik's in that he uses many

of Frolik's ideas and is able to extend a few of Frolik's results. Using Frolik's notation, let $\mathcal{D}$ denote the class of all completely regular spaces X such that $X \times Y$ is pseudocompact for every completely regular, pseudocompact space Y. Frolik was able to show that the product of two spaces from $\mathcal{D}$ is again in $\mathcal{D}$. Noble was able to extend this theorem to arbitrary products.

Before proving this theorem, we need a lemma. Let $\mathcal{P}$ represent the following condition: for each filter Φ consisting of infinite subsets of N, $\bigcap_{N' \in \Phi} \overline{\bigcup_{n \in N'} U_n} \neq \phi$.

<u>Lemma 13.13</u>. Let $\{U_n \mid n \in N\}$ be a family of subsets of X.

(a) If $\{U_n\}$ satisfies condition $\mathcal{P}$, if S is any finite collection of subsets of X, and if $\{S_n\}$ is any indexing of $\{U_n \cup S\}$ by N, then $\{S_n\}$ satisfies the condition $\mathcal{P}$.

(b) If X is in $\mathcal{D}$ and each U_n is non-empty and open, then for some infinite $N' \subseteq N$, $\{U_n \mid n \in N'\}$ satisfies the condition $\mathcal{P}$.

<u>Proof</u>: (a) This is obvious.

(b) Choose $x_1 \in U_1 = U_{n_1}$. If each neighborhood of x_1 meets all but finitely many of the U_n, we are done. If not, then there exists an open neighborhood $V_1 \subseteq U_1$ of x and an infinite subset N_1 of N such that $V_1 \cap U_n = \phi$ for each $n \in N_1$. Proceeding inductively, choose $n_k \in N_{k-1}$ and $x_k \in U_{n_k}$. Either each neighborhood of x_k meets all but

finitely many of the sets U_n with $n \in N_{k-1}$, or there exists an open neighborhood $V_k \subseteq U_{n_k}$ of x_k and an infinite subset N_k of N_{k-1} such that $V_k \cap U_n = \phi$ for each $n \in N_k$. If the induction continues, we obtain an infinite disjoint family $\{V_n\}$ of non-empty sets. Since X is in $\mathcal{D}$, some infinite subfamily of $\{V_n\}$ and hence some infinite subfamily of $\{U_{n_k}\}$ must satisfy $\mathcal{P}$.

Theorem 13.14. The class $\mathcal{D}$ is closed under arbitrary products.

Proof: We have previously shown that a product space is pseudocompact if and only if each of its countable subproducts is pseudocompact (Theorem 13.2). Therefore a product space is in $\mathcal{D}$ if and only if each of its countable subproducts is in $\mathcal{D}$. Then it suffices to consider a countable product, say $X = \prod_i X_i$ of members from $\mathcal{D}$.

Let $\mathcal{F}$ be an infinite collection of non-empty open subsets of X . We may suppose that $\mathcal{F} = \{U_n\}$ where each $U_n = \prod_i U_n^i$ with every U_n^i open, and for each n , all but finitely many of them equal to X_i .

Using part (b) of Lemma 13.13, choose infinite subsets N_i of N with $N_i \subseteq N_{i-1}$ such that $\{U_n^i \mid n \in N_i\}$ satisfies the condition $\mathcal{P}$. Choose $n_i \in N_i$ and, using part (b) of Lemma 13.13, assume that for $j < i$, n_j is in N_i . To see that $\{U_{n_i} \mid i \in N\}$ satisfies condition $\mathcal{P}$, let Φ be any filter consisting of infinite subsets of N and choose, inductively, $x_k \in X_k$ so that $(x_1,\cdots,x_k)$ is the closure of $\bigcup_{j \in N'} \prod_{i=1}^{k} U_{n_j}^i$ for each $N' \in \Phi$. (This can be done by part (a) of Lemma 13.13.) Then for

$x = (x_1, \cdots, x_k, \cdots)$, x is in the closure of $\bigcup_{j \in N'} U_{n_j}$ for each $N' \in \Phi$. Therefore, we're done by Theorem 13.9.

Let $\mathcal{D}^*$ denote the class of all spaces X with the following property: each infinite collection of disjoint open sets has an infinite subcollection each of which meets some fixed compact set. Frolik was able to show that any space satisfying this condition belongs to $\mathcal{D}$ (Theorem 13.8). Hence $\mathcal{D}^* \subset \mathcal{D}$.

Noble was able to characterize some of the spaces which belong to the class $\mathcal{D}^*$. A space X is said to be a k_R - SPACE if it has the following property: a real valued function with domain X is continuous if its restriction to each compact subset of X is continuous. As with k - spaces, associated with each space X there is a unique k_R - space, say $k_R X$, having the same underlying set and the same compact subsets as X .

Theorem 13.15. A k_R - space is pseudocompact if and only if it is in $\mathcal{D}^*$. Thus if $k_R X$ is pseudocompact, then X is in $\mathcal{D}^*$.

Proof: $\longleftarrow$. It is clear that any space in $\mathcal{D}^*$ is pseudocompact.

$\longrightarrow$. Suppose X is a k_R - space which is not in $\mathcal{D}^*$. Let $\{U_n\}$ be a countable collection of disjoint open subsets of X such that for each compact $K \subseteq X$, $K \cap U_n = \phi$ with only finitely many exceptions. For each n , let f_n be a continuous real-valued function which maps $X - U_n$ to 0 and some point in U_n to n . Set $f = \sum_n f_n$. Then f is

continuous on compact sets and hence continuous. But f is unbounded, so X is not pseudocompact which is a contradiction.

§14. Perfect Normality. A space X is said to be PERFECTLY NORMAL if it is normal and each closed subset in X is a G_δ. Equivalently, X is perfectly normal if it is normal and each open subset in X is an F_σ. The product of two perfectly normal spaces need not be perfectly normal. The Sorgenfrey line (§1.1) is an example of a perfectly normal space whose product with itself is not perfectly normal.

Morita was able to determine that it is sufficient for a space to be perfectly normal in order that the product space $X \times Y$ be perfectly normal for every metric space Y. The following lemma is needed.

Lemma 14.1. Every perfectly normal space is a P - space.

Proof: Every open subset of a perfectly normal space is an F_σ subset, so this result follows immediately from Lemma 2.1.

Theorem 14.2: If X is a perfectly normal space, then $X \times Y$ is perfectly normal for every metric space Y.

Proof: By Lemma 14.1, we have that X must be a P - space. Then X is a normal P - space which implies that $X \times Y$ is normal by Theorem 2.18. We must show that any open subset of $X \times Y$ is an F_σ.

Suppose G is any open subset of $X \times Y$. Let $\{V_\alpha \mid \alpha \in \Omega\}$ be a σ - locally finite open basis of Y. Then there exist open subsets

H_α , $\alpha \in \Omega$ of X such that $G = \cup\{H_\alpha \times V_\alpha \mid \alpha \in \Omega\}$. Since each $H_\alpha \times V_\alpha$ is an F_σ and $\{H_\alpha \times V_\alpha \mid \alpha \in \Omega\}$ is a σ - locally finite collection in $X \times Y$, it follows that G is an F_σ .

BIBLIOGRAPHY

[Do] Dowker, C.H. "On Countably Paracompact Spaces", Can. J. Math 1, 219-224 (1951).

[Du] Dugundji. Topology. Allyn and Bacon, Inc., Boston (1966).

[F_1] Frolik, Z. "On the Topological Product of Paracompact Spaces". Bull. Acad. Polon. Sci. 8, 747-750 (1960).

[F_2] Frolik, Z. "The Topological Product of Countably Compact Spaces". Czech. Math. J. 10, 329-338 (1960).

[F_3] Frolik, Z. "The Topological Product of Two Pseudocompact Spaces". Czech. Math. J. 10, 339-348 (1960).

[GJ] Gillman, L., and Jerison, M. Rings of Continuous Functions. Van Nostrand Company Ltd., Toronto (1960).

[G] Glicksberg, I. "Stone-Cech Compactifications of Products". Trans. Am. Math. Soc. 90, 369-382 (1959).

[I] Ishii, T. "On Product Spaces and Product Mappings". J. Math. Soc. Japan 18, 166-181 (1966).

[Kate] Katetov, M. "Complete Normality of Cartesian Products". Fund. Math. 35, 271-274 (1948).

[$Katu_1$] Katuta, Y. "A Theorem on Paracompactness of Product Spaces". Proc. Japan Acad. 43, 615-618 (1967).

[$Katu_2$] Katuta, Y. "On the Normality of the Product of a Normal Space with a Paracompact Space". Gen. Top. and its Appl. 1, 295-319 (1971).

[Mi_1] Michael, E. "A Note on Paracompact Spaces". Proc. Am. Math. Soc. 4, 831-838 (1953).

[Mi_2] Michael, E. "Paracompactness and the Lindelof Property in Finite and Countable Cartesian Products". Comp. Math. 23, 199-214 (1971).

[Mo_1] Morita, K. "A Condition for Metrizability of Topological Spaces and for n-Dimensionality". Sci. Reports Tokyo Kyoiku Daigaku, Sect. A, 5, 33-36 (1955).

[Mo_2] Morita, K. "Note on Paracompactness". Proc. Japan Acad. 37, 1-3 (1961).

[Mo_3] Morita, K. "Paracompactness and Product Spaces". Fund. Math. 50, 223-236 (1962).

[Mo_4] Morita, K. "On the Product of a Normal Space with a Metric Space". Proc. Japan Acad. 39, 148-150 (1963).

[Mo_5] Morita, K. "On the Product of Paracompact Spaces". Proc. Japan Acad. 39, 559-563 (1963).

[Mo_6] Morita, K. "Products of Normal Spaces with Metric Spaces I". Math. Ann. 165, 365-382 (1964).

[Mo_7] Morita, K. "Products of Normal Spaces with Metric Spaces II". Sci. Reports Toyko Kyoiku Daigaku 8, 1-6 (1963).

[N] Noble, N. "Countably Compact and Pseudocompact Products". Czech. Math. J. 19, 390-397 (1969).

[R] Rudin, M.E. "A Normal Space X for which $X \times I$ is not Normal". Fund. Math. 73, 179-186 (1971).

[So] Sorgenfrey, R.H. "On the Topological Product of Paracompact Spaces". Bull. Am. Math. Soc. 53, 631-632 (1947).

[St] Stone, A.H. "Paracompactness and Product Spaces". Bull. Am. Math. Soc. 54, 977-982 (1948).

[Su] Suzuki, J. "Paracompactness and Product Spaces" Proc. Japan Acad. 45, 457-460 (1969).

[Ta_1] Tamano, H. "On Paracompactness". Pac. J. Math 10, 1043-1047 (1960).

[Ta_2] Tamano, H. "On Compactifications". J. Math. Kyoto Univ. 1, 162-193 (1962).

[Ta_3] Tamano, H. "Note on Paracompactness". J. Math. Kyoto Univ. 3, 137-143 (1963).

[Ta_4] Tamano, H. "Normality and Product Spaces". Gen. Top. and Its Relation to Modern Analysis and Algebra II. (Proc. Sympos., Prague, 1966), Academic Press, New York (1967), 349-352.

[Ts] Tsuda, M. "On the Normality of Certain Product Spaces". Proc. Japan Acad. 40, 465-467 (1964).

[W] Willard, S. General Topology. Addison-Wesley (Canada) Ltd., Don Mills (1970).